4.1 Work Done by a Constant Force

Physics Tool box

- **Work (W)** – the energy transferred to or from an object by means of a force acting on the object.
- **Positive Work** – the work done when the displacement and the applied force act in the same direction.
- **Negative Work** - the work done when the displacement and the applied force act in opposite directions.
- **Joule (J)** – SI derived unit for measuring forms of work and energy; equal to the work done when a force of 1 N displaces an object 1 m in the direction of the force.
- $W = \vec{F} \cdot \vec{d}$
- $W = Fd\cos(\phi)$
- Work = area under a **Force** vs **Change in Distance** graph
- Zero displacement = zero work

In everyday language, the term "work" has many meanings, however in physics work is the energy transferred to an object. The term transfer can be misleading. It does not mean that anything material flows into or out of the object; that is the transfer is not like a flow of water. Rather, it is like the electronic transfer of money between two bank accounts. The number in one account goes up while the number in the other account goes down, with nothing material passing between the two accounts.

To calculate the work a force does on an object as the object moves through some displacement, we use only the force component along the objects displacement. The force component perpendicular to the displacement does zero work.

Mathematically we say $W = \vec{F} \cdot \vec{d}$ (work done by a constant force), where we have the dot product between the two vectors.

The resulting mathematical definition can be written as $W = Fd\cos(\phi)$, where ϕ is the angle between the directions of the force and the displacement.

Example

How much work is done by you pulling a wagon with a horizontal force of 50 N for a distance of 6.4 m?

Solution:

Given: $\vec{F}_{app} = 50N$, $\Delta\vec{d} = 6.4m$, W=?

$$\begin{aligned} W &= \vec{F}_{app}\Delta\vec{d} \\ &= (50N)(6.4m) \\ &= 320J \end{aligned}$$

Example

A student is pulling a wagon at a constant velocity with a force of 60N. The handle of the wagon is at $33°$ above the horizontal. How much work is done by the student if she must move the wagon 0.25 km?

Solution:

First, we must convert the displacement into metres.

$$\Delta d = 0.25km\left(\frac{1000m}{1km}\right) = 250m$$

The force used in this calculation must be the component of the applied force in the direction of motion

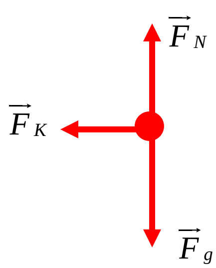
$$\begin{aligned} W &= F_{app}\left(\cos(\phi)\right)\Delta d \\ &= 60N\cos(33°)250m \\ &= 1.3\times10^4 J \end{aligned}$$

Example

You toboggan down a hill and come to a stop 20.1 m along the horizontal at the bottom of the hill. The total mass of you and the toboggan is 73.2 kg, and the coefficient of kinetic friction between the toboggan and the snow is 0.106.

a) Draw a free body diagram of the situation and determine the magnitude of Kinetic Friction.
b) Calculate the work done by the friction bringing the toboggan to a stop.

Solution:



Given: $m = 73.2kg$, $\left|\vec{F}_N\right| = \left|m\vec{g}\right|$, $\mu_K = 0.106$

Now, $F_K = \mu_K F_N$

$$\begin{aligned} F_K &= \mu_K F_N \\ &= \mu_K(mg) \\ &= (0.106)(73.2kg)\left(9.80\frac{N}{kg}\right) \\ &\approx 76.0N \end{aligned}$$

Units of g are also N/kg

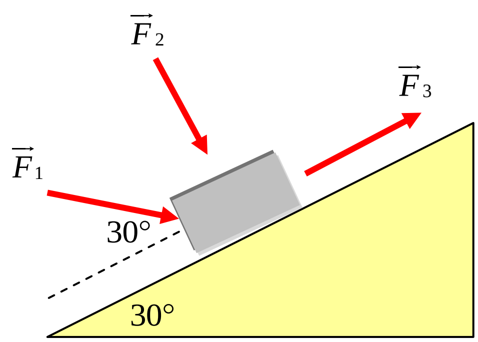
b)

$$W = F_K \cos(\theta)\Delta d$$
$$= 76.0N\cos(180°)(20.1m)$$
$$= -1.53\times 10^3 J$$

Because the work done by Kinetic Friction is negative, it causes a decrease in speed.

Example

A block moves up a $30°$ incline under the action of three forces. $\vec{F}_1$ is horizontal and of magnitude 40N. $\vec{F}_2$ is normal to the plane and of magnitude 20N. $\vec{F}_3$ is parallel to the plane and of magnitude 30N. Determine the work done by each force as the block moves 80cm up the incline.



Solution:

Displacement is 0.8m

$\vec{F}_1$ component:

The component of $\vec{F}_1$ along the direction of the displacement is:

$$F_1\cos(30°) = (40N)(0.866) = 34.6N$$

The work done by $\vec{F}_1$ is $W = F_1 d = (34.6N)(0.80m) = 28J$

$\vec{F}_2$ component:

Because it has no component in the direction of the displacement, F_2 does no work.

$\vec{F}_3$ component:

The component of $\vec{F}_3$ in the direction of displacement is 30N, hence the work done by $\vec{F}_3$ is $W = F_1 d = (30N)(0.80m) = 24J$

Example

A ladder 4.0m long and weighing 300N has its centre of gravity 100 cm from the bottom. At its top end is a 50N weight. Compute the work required to raise the ladder from a horizontal position on the ground to a vertical position.

Solution:

The work done (against gravity) consists of two parts, the work to raise the centre of gravity 1 m and the work to raise the weight at the end through 4 m. Therefore

Work done = Centre of gravity + raising end =(300N)(1.00m) + (50N)(4.0 m)=500J

Therefore the total work done is 500J

4.2 Kinetic Energy and the Work-Energy Theorem

Physics Tool box

- **Kinetic Energy (E_K)** – energy of motion.
- **Work energy Theorem** – the total work done on an object equals the change in the object's kinetic energy, provided there is no change in any other form of energy (ie gravitational potential energy).
- $E_K = \frac{1}{2}mv^2$
- $W = \Delta E$
- $W_{total} = E_{K_f} - E_{K_i} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$
- $1J = 1N \cdot m = 1\left(kg \cdot \frac{m}{s^2}\right) \cdot m = 1kg \cdot \frac{m^2}{s^2}$

Calculus: Work and Energy with varying force

- $W = \int_{x_1}^{x_2} F_x dx$

Calculus: Work done on a curved path

- $W = \int_{p_1}^{p_2} F\cos(\phi)dl = \int_{p_1}^{p_2} \vec{F} \cdot d\vec{l}$

The total work done on a body by external forces is related to the body's displacement – that is, to changes in its position.

Simply $W_{total} = \left(\sum F\right)\Delta d$

But we know from Newton's Second Law that $\sum F = ma$, now with a constant force F, then the acceleration a, is constant.

We also know that $v_f^2 = v_i^2 + 2a\Delta d$ or $a = \frac{v_f^2 - v_i^2}{2\Delta d}$, therefore

$$\begin{aligned}
W_{total} &= \left(\sum F\right)\Delta d \\
&= (ma)\Delta d \\
&= m\left(\frac{v_f^2 - v_i^2}{2\Delta d}\right)\Delta d \\
&= m\left(\frac{v_f^2 - v_i^2}{2}\right) \\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2
\end{aligned}$$

Example

What is the total work, in megajoules, that is required to cause the space Shuttle of mass $1.05\times10^5\,kg$ to increase its speed in level flight from 500 m/s to 700 m/s?

Solution:

Given: m= $1.05\times10^5\,kg$, $v_f = 700\dfrac{m}{s}$, $v_i = 500\dfrac{m}{s}$, $W_{total} = ?$

$$
\begin{aligned}
W_{total} &= \Delta E_K \\
&= E_{K_f} - E_{K_i} \\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \\
&= \frac{1}{2}1.05\times10^5\,kg\left(700\frac{m}{s}\right)^2 - \frac{1}{2}1.05\times10^5\,kg\left(500\frac{m}{s}\right)^2 \\
&= 1.26\times10^{10}\,kg\cdot\frac{m^2}{s^2} \\
&= 1.26\times10^{10}\,J \\
&= 1.26\times10^4\,MJ
\end{aligned}
$$

Example

A truck of mass $2.3\times10^4 kg$, was traveling down the 401 at some initial speed, has $-3.1MJ$ of work done on it, causing its speed to become $30\dfrac{km}{h}$. Determine its initial speed.

Solution:

Negative total work indicates that the object's kinetic energy decreases (that is, its speed decreases).

First lets put everything into correct units

$$-3.1MJ = -3.1\times10^6\,J$$

$$30\frac{km}{h} = 30\frac{km}{h}\times\frac{1000m}{1km}\times\frac{1h}{3600s} = 8.3\frac{m}{s}$$

Now, We are given: $m = 2.3\times10^4 kg$, $\Delta E_K = -3.1\times10^6 J$, $v_f = 8.3\frac{m}{s}$, $v_i = ?$

$$\begin{aligned}
\Delta E_K &= E_{K_f} - E_{K_i} \\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \\
2\Delta E_K &= mv_f^2 - mv_i^2 \\
mv_i^2 &= mv_f^2 - 2\Delta E_K \\
v_i^2 &= \frac{mv_f^2 - 2\Delta E_K}{m} \\
v_i &= \pm\sqrt{\frac{mv_f^2 - 2\Delta E_K}{m}} \\
&= \pm\sqrt{\frac{\left(2.3\times10^4 kg\right)\left(8.3\frac{m}{s}\right)^2 - 2\left(-3.1\times10^6 J\right)}{2.3\times10^4 kg}} \\
&= \pm18.397\frac{m}{s} \\
&= 18\frac{m}{s}
\end{aligned}$$

We choose the positive value because speed is always positive.

The initial speed was then $18\frac{m}{s}\times\frac{1km}{1000m}\times\frac{3600s}{1h} = 64.8\frac{km}{h}$

The initial speed was about 65 km/h

4.4 Conservation of Energy

Physics Tool box

- **Law of Conservation of Energy** – for an isolated system, energy can be converted into different forms, but cannot be created or destroyed.
- **Isolated System** – a system of particles that is completely isolated from outside influences.
- **Thermal Energy (E_{th})** – internal energy associated with the motion of atoms and molecules.
- $E_{total_{inital}} = E_{total_{final}}$
- $W = \Delta E_g = mg\Delta h$
- $\frac{1}{2}mv_1^2 + mg\Delta h_1 = \frac{1}{2}mv_2^2 + mg\Delta h_2$
- $E_{th} = F_K \Delta d$, recall F_K is the magnitude of Kinetic Friction

The law of conservation of energy cannot (as we know it) be violated. It is one of the fundamental principles in operation in the universe. It allows the combination of kinetic energy and potential energy (gravitational) to be combined to provide the total energy (mechanical) in the system. The energy cannot be lost, just converted to some other form.

The knowledge of this law, $E_{T_f} = E_{T_i}$, can allow us to solve some interesting problems.

Example

A 3.0 kg object rolls across then down an incline 30 m high at 6.0 m/s, then rolls up a ramp inclined at $30°$.

a) Find the speed at the bottom of the incline.
b) Find the height reached on the other ramp.
c) Find the distance travelled up the ramp.
d) Find the acceleration along the ramp (can you do two ways).

Solution:

a) Given: $m = 3.0kg$, $v = 6.0\frac{m}{s}$, $\Delta h = 30m$, $\theta = 30°$

$$E_{T_i} = E_{p_i} + E_{k_i}$$
$$= mg\Delta h + \frac{1}{2}mv^2$$
$$= (3.0kg)\left(9.8\frac{m}{s^2}\right)(30m) + \frac{1}{2}(3.0kg)\left(6.0\frac{m}{s}\right)^2$$
$$= 936\,J$$

To find the speed at the bottom of the ramp,

$E_{T_f} = E_k$, since no potential energy

Now $E_{T_i} = E_{T_f}$

$$936\,J = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2(936J)}{3.0kg}}$$
$$= 24.97999\frac{m}{s} \approx 25\frac{m}{s}$$

b) since when the ball goes up the ramp, it must stop at its highest point when

$$E_T = E_p$$
$$936J = mg\Delta h$$
$$\Delta h = \frac{936J}{(3.0kg)\left(9.8\frac{m}{s^2}\right)}$$
$$= 31.8367m \approx 32m$$

The object reached the height of 32m on the ramp

c) $\sin(\theta) = \frac{\Delta h}{\Delta d}$

$$\Delta d = \frac{31.8367m}{\sin(30°)}$$
$$= 63.6734m \approx 64m$$

The distance travelled up the ramp is 64m d)

Method 1: $a = g\sin(\theta)$

$$= \left(-9.8\frac{m}{s^2}\right)\sin(30°)$$

$$= -4.9\frac{m}{s^2}$$

Method 2: $v_f^2 = v_i^2 + 2a(\Delta d)$

$$a = \frac{v_f^2 - v_i^2}{2(\Delta d)}$$

$$= \frac{0 - 624\frac{m^2}{s^2}}{127.34m}$$

$$= -4.9\frac{m}{s^2}$$

Example

You flick a 1g coin along the table top and it slides across the table. If the coin travels along a straight line of 1.20m, experiencing friction with a coefficient of kinetic friction 0f $\mu_K = 0.611$.

a) How much thermal energy is produced during the slide?
b) Determine, the coins speed just after it was flicked.

Solution:

a) Given: $\mu_K = 0.611,\ m = 0.001kg,\ \Delta d = 1.20m$

We know: $F_K = u_K F_N,\ F_N = mg$

$$E_{th} = F_K \Delta d$$
$$= u_K F_N \Delta d$$
$$= u_K mg\Delta d$$
$$= (0.611)(0.001kg)\left(9.8\frac{N}{kg}\right)(1.20m)$$
$$= 7.19\times10^{-3}J$$

The thermal energy produced is $7.19\times10^{-3}J$

b) Since the coin comes to a complete stop and $E_{T_i} = E_{T_f}$. Then $E_{T_i} = E_{th}$

$$E_{T_i} = E_{th}$$

$$\frac{1}{2}mv_i^2 = E_{th}$$

$$v_i^2 = \frac{2E_{th}}{m}$$

$$v_i = \pm\sqrt{\frac{2\left(7.19\times10^{-3}\,J\right)}{0.001kg}}$$

$$= 3.79\frac{m}{s}$$

4.5 Hooke's Law

Physics Tool box

- **Hooke's Law** – the magnitude of the force exerted by a spring is directly proportional to the distance the spring has moved from equilibrium.
- **Ideal Spring** – a spring that obeys Hooke's law because it experiences no internal or external friction.
- **Force constant (k)** – the proportionality constant of a spring measured in newtons per metre.
- **Elastic Potential Energy (E_e)** – the energy stored in an object that is stretched, compressed, bent, or twisted.
- **Equilibrium Position** – the position of the mass when the spring is neither stretched nor compressed, since there is zero acceleration and zero total force on m at this location.
- The elastic potential energy stored in a spring is proportional to the force constant of the spring and to the square of the stretch or compression.
- $F_x = kx$ (force required to stretch or compress a spring)
- $F_x = -kx$ (force exerted by the spring)

Whether you pull on an expansion spring or push on a compression spring, the effect is always the same; the spring tries to restore itself to its original length.

The minus sign in $F_x = -kx$ indicates that the direction of the spring force is always opposite the direction of the displacement of the spring's free end.

k is called the **spring constant** (or **force constant**) and is a measure of the stiffness of the spring. The larger **k** is, the stiffer the spring. The SI unit for **k** is the Newton per metre.

Example

You stretch a spring horizontally a distance of 24 mm by applying a force of 0.21N [E].

a) Determine the force constant k of the spring.
b) What is the force exerted by the spring on you?

Solution:

a) $F_x = 0.21N, x = 0.024m,\ k = ?$

Since the force is applied to the spring, we use the equation

$$F_x = kx$$

$$k = \frac{F_x}{x} = \frac{0.21N}{0.024m} = 8.75\frac{N}{m}$$

b) Since Newton's third law, states that is the force applied to the spring is0.21N [E], then the force exerted by the spring is 0.21 N [W]

Example

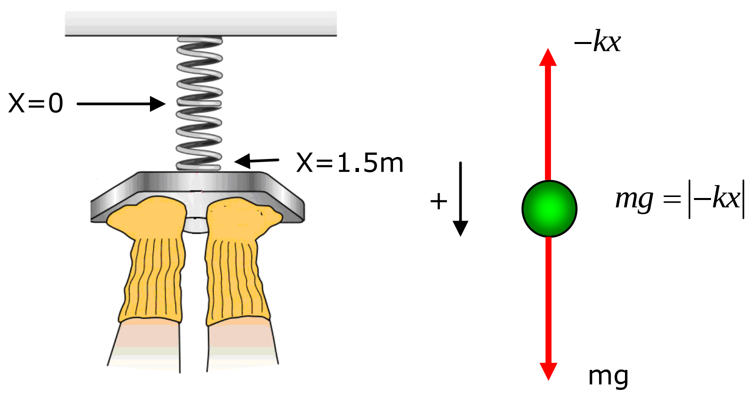
A contestant on Fear Factor has mass of 60 kg is hanging from a spring that has stretched to a new length and now is in a equilibrium position 1.5m below the springs initial equilibrium position.

a) Determine the force constant of the spring.

b) if the contestant is pushed up to the spring's unstretched equilibrium position and then allowed to fall, what is the net force on the contestant when it has dropped 0.5m?

c) Determine the acceleration of the contestant at the position in b)

Solution:



If we choose + to be downward, will x=0 representing the position of the unstretched spring. We have two vertical forces acting of the contestant: gravity in the downward direction and the upward force of the spring. At the new equilibrium the contestant is stationary, so the net force on it is zero.

Now $m = 60kg, x = 1.5m, k = ?$

$$\sum F_y = 0$$

$$mg + (-kx) = 0$$

$$k = \frac{mg}{x}$$

$$= \frac{(60kg)\left(9.8\frac{N}{kg}\right)}{1.5m}$$

$$= 392\frac{N}{m}$$

b) $\sum F_y = mg + (-kx) = (60kg)\left(9.8\frac{N}{kg}\right) - \left(392\frac{N}{m}\right)(0.5m) = 392N$

The force on the contestant is 392 N [down]

c) $\sum F = ma$

$$a = \frac{392N}{60kg}$$

$$= 6.53333\frac{N}{kg}$$

$$= 6.5\frac{m}{s^2}$$

4.6 Elastic Potential Energy

Physics Tool box

- **Force constant (k)** – the proportionality constant of a spring measured in newtons per metre.
- **Elastic Potential Energy (E_e)** – the energy stored in an object that is stretched, compressed, bent, or twisted.
- **Equilibrium Position** – the position of the mass when the spring is neither stretched nor compressed, since there is zero acceleration and zero total force on m at this location.
- The elastic potential energy stored in a spring is proportional to the force constant of the spring and to the square of the stretch or compression.
- **Simple Harmonic Motion (SHM)** – is the periodic vibratory motion such that the force (and thus the acceleration) is directly proportional to the displacement.
- A reference circle can be used to derive equations for the period and frequency of SHM.
- **Damped Harmonic Motion** – is periodic motion in which the amplitude of vibration and energy decrease with time.
- $F_x = kx$ (force required to stretch or compress a spring)
- $F_x = -kx$ (force exerted by the spring)
- $E_e = \frac{1}{2}kx^2$ (elastic potential energy)
- $W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$ (work done **by** a spring),
- $W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$ (work done **on** a spring)

When a railroad car runs into a spring bumper at the end of a track, the spring is compressed as the railcard is brought to a stop. The bumper springs back and the railroad car moves away in the opposite direction with its original speed. During the interaction with the spring, the railcars kinetic energy has been "**stored**" in the elastic deformation of the spring.

Something similar happens with a rubber-band. Stretching the rubber-band "**stores**" energy, this can later be converted to kinetic energy. We describe the process of storing energy in a deformable body (such as a rubber-band or a spring) in terms of **elastic potential energy**. A body is called **elastic** if it returns to its original shape and size after being deformed.

The work done in compressing a spring is $W = \frac{1}{2}F_x x = \frac{1}{2}(kx)x = \frac{1}{2}kx^2$

Since the work has be converted (stored) as elastic potential energy, we can rewrite the Work equation as:

$$E_e = \frac{1}{2}kx^2$$

Example

A 2.2 kg mass is attached to the end of a vertical spring (the spring has a force constant of 10.7 N/m). The mass is held so that the spring is a a unstretched equilibrium position, then it is allowed to fall. Neglecting the mass of the spring, determine

a) How much elastic potential energy is stored in the spring as the mass falls 20 cm

b) the speed of the mass when it has fallen 20 cm

Solution:

We will let the x position of the unstretched spring as 0 and will choose downward as +, and Δh represent the difference in gravitational potential energy heights

a) $x = 20cm = 0.20m, \Delta h = 0.20m, k = 10.7\frac{N}{m},\ E_e = ?$

$$\begin{aligned} E_e &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}\left(10.7\frac{N}{m}\right)(0.20m)^2 \\ &= 0.214J \end{aligned}$$

The elastic potential energy stored in the spring is 0.214 J

b) We must apply the law of conservation of energy to solve this problem

$$E_{T_i} = E_{T_f}$$

$$E_{g_i} + E_{k_i} + E_{e_i} = E_{g_f} + E_{k_f} + E_{e_f}$$

No initial movement therefore no kinetic energy, and since no stretch we also have no elastic potential energy,

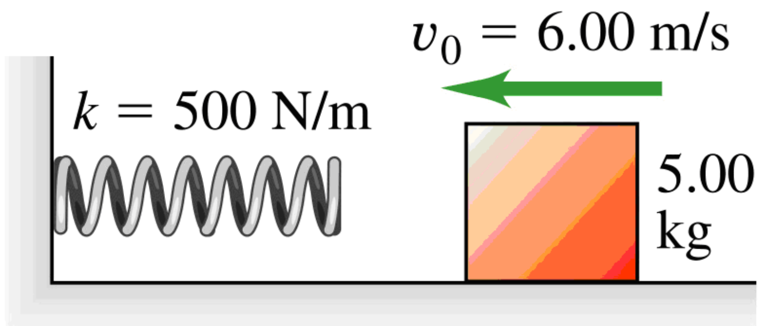
$$mgh_i + 0 + 0 = mgh_f + \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\begin{aligned} mgh_i - mgh_f &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ mg(\Delta h) &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ \frac{1}{2}mv^2 &= mg(\Delta h) - \frac{1}{2}kx^2 \\ v &= \pm\sqrt{2g(\Delta h) - \frac{kx^2}{m}} \\ &= \pm\sqrt{2\left(9.8\frac{m}{s^2}\right)(0.20m) - \frac{10.7\frac{N}{m}(0.20m)^2}{2.2kg}} \\ &= \pm 1.930\frac{m}{s} \end{aligned}$$

The speed of the mass is 1.9 m/s (since speed is always positive)

Example

A 5.00 kg block is moving at $v_0 = 6.00\frac{m}{s}$ along a frictionless, horizontal surface towards a spring (the spring has a force constant of $500\frac{N}{m}$) that is attached to a wall. Assuming the spring has negligible mass, determine the distance the spring will be compressed.



Solution:

At the point of maximum compression, the spring (and therefore the block) is not moving, so the block has no kinetic energy. This is transferred to the spring (via the work done on the spring by lowing the block) which stores is as elastic potential energy.

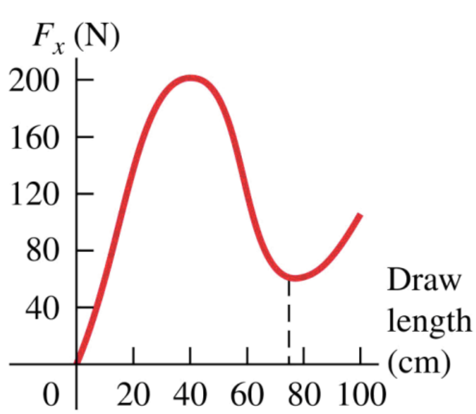
$$E_e = E_k$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$x = v\sqrt{\frac{m}{k}} = \left(6.00\frac{m}{s}\right)\sqrt{\frac{5.00kg}{500\frac{N}{m}}} = 0.600m$$

Example

The figure below shows how the force exerted by the string of a compound bow on an arrow varies as a function of how far back the arrow is pulled (the **draw length**). Assume the same force is exerted on the arrow as it moves forward after being released. Full draw for this for this bow is at a draw length of 75.0 cm. If the bow shoots a 0.0250 kg arrow from full draw, what is the speed of the arrow as it leaves the bow?



Solution:

The arrow will acquire the energy that was used in drawing the bow (the work done by the archer), which will be the area under the curve that represents the force as a function of distance

Approximate the area under the curve to be from (76 to 100)J

Therefore $W = E_e = E_k = \frac{1}{2}mv^2$

$$100J = \frac{1}{2}(0.025kg)v^2$$

$$v = \sqrt{\frac{200J}{0.025kg}}$$

$$= 89\frac{m}{s}$$

The speed of the arrow is a bout 89 m/s

4.8 Wave Equation Review

Physics Tool box

- **Equation:** $\Psi(x,t) = A\cos(kx \pm \omega t + \phi)$
- A is the amplitude
- ω (omega) is the angular frequency and is defined as 2π times the frequency (cycles per sec): $\omega = 2\pi f$.
- k is called the angular wave number (it is equal to 2π times the wave number, the angular wave number represents the number of wave-lengths in a distance of 2π)
- ϕ is the phase

The location of a point in a medium that conducts a wave depends on two variables, position (x) and time (t). It has the basic form .

$$\Psi(x,t) = A\cos(kx \pm \omega t + \phi)$$

To show that y depends on both x and t, the dependent variable Ψ , is often written as $\Psi(x,t)$. If the minus sign is used in the equation above, the wave travels in the +x direction, if the plus sign is used, the wave travels in the –x direction.

What do the coefficient A, k, ω, ϕ represent?

- A is the amplitude
- ω (omega) is the angular frequency and is defined as 2π times the frequency (cycles per sec): $\omega = 2\pi f$.
- k is called the angular wave number (it is equal to 2π times the wave number, the angular wave number represents the number of wave-lengths in a distance of 2π)
- ϕ is the phase

We will not be interested in the phase in this unit

Wave properties:

$$k = \frac{2\pi}{\lambda} \Leftrightarrow \lambda = \frac{2\pi}{k}$$

$f = \frac{1}{T}$ (T is the period of the wave)

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$v = \lambda f$$

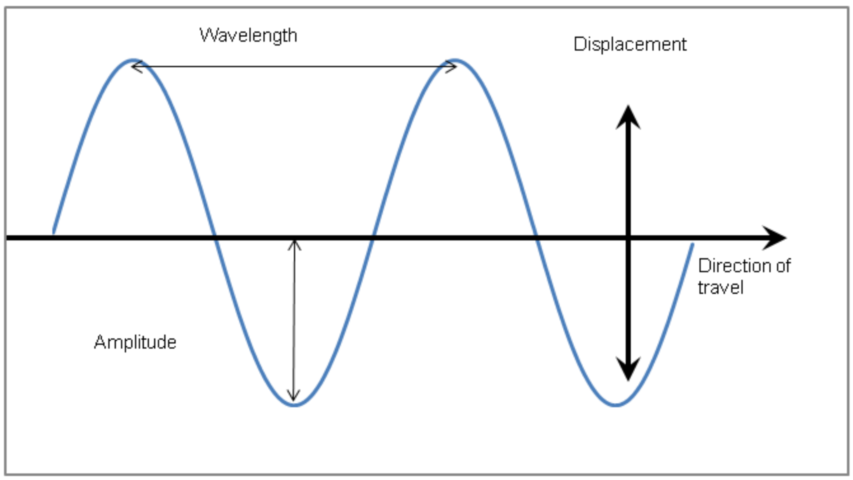
Transverse traveling waves

Instead of looking at a wave in which both variables (x and t) are changing, we will look at a wave and allow only one of the variables to change. That is, we will use two points of view:

Point of view 1: x varies, t does not
Point of view 2: t varies , x does not

Point of View 1 (x varies, t does not)

In order to keep t from changing, we must freeze time. We do this by imaging a still photograph of the wave at some fixed time, t. All of the x positions of the wave are visible, and we can determine some properties of the wave

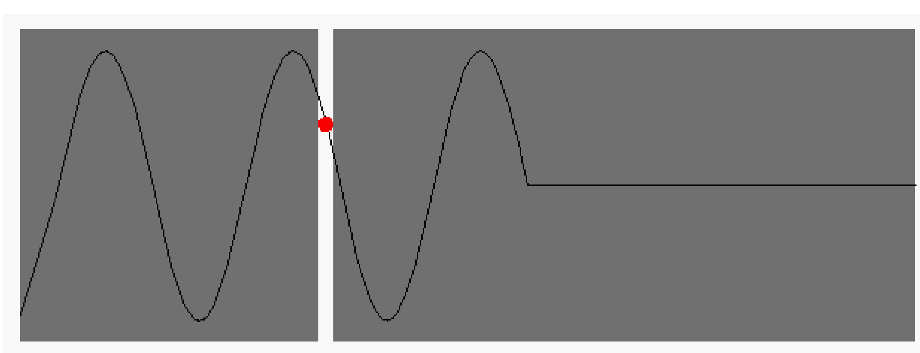


Because we have the horizontal axis (the x-axis), we can determine the distance between crests, called the wavelength. And the maximum displacement from the horizontal equilibrium position (the amplitude).

Point of View 2 (t varies , x does not)

Now we will designate one position x along the wave, and watch how this position changes as time varies.

What we are interested in (and thus we will plot) is the single point which will move up and down as time varies (see applet). The time for the point to move up and down is called its period T, and the number of cycles it completes in one second is called its frequency, f.



Summary

Four of the most important characteristic of any wave are its wavelength, amplitude, period, and frequency.

The first two of these were identified in Point of View 1, and the second two were identified in point of view 2.

A fifth characteristic of any wave is its speed which is $v = \lambda f$

Example

Given $\Psi(x,t) = 0.4\cos(6x - 10t)$
Where x is expressed in metres and t is in seconds

Determine

a) amplitude
b) wave length
c) frequency
d) speed
e) $\Psi(5,2)$
f) angular frequency
g) angular wave number
h) wave number
i) Period

Solution:

a) Amplitude is 0.4 m

b) wave length is $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{6} = \dfrac{\pi}{3} m \approx 1.05 m$

c) frequency is $f = \dfrac{\omega}{2\pi} = \dfrac{10}{2\pi} = \dfrac{5}{\pi} \approx 1.59\,Hz$

d) speed is $v = f\lambda = (1.59\,Hz)(1.05\,m) = 1.67\dfrac{m}{s}$

e) $\Psi(5,2) = -0.34\,m$

f) angular frequency is $\omega = 10\dfrac{rad}{s}$

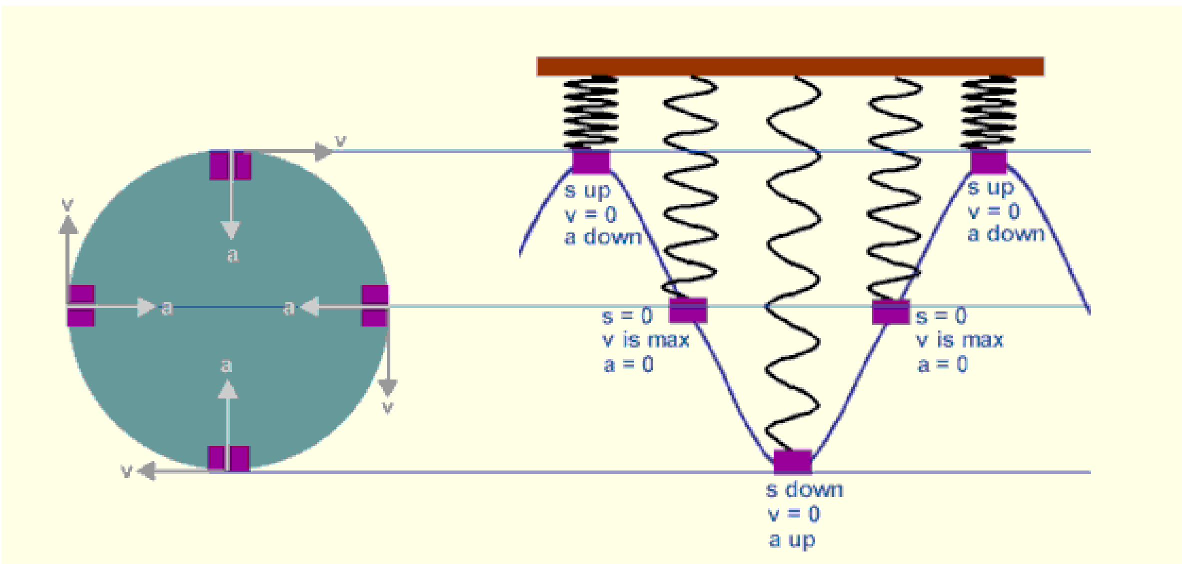
g) angular wave number is $6\dfrac{rad}{m}$

h) wave number $\dfrac{6}{2\pi} m^{-1}$

i) Period is $T = \frac{1}{f} = \frac{1}{5/\pi} = \frac{\pi}{5} s$

Harmonic oscillators

On a horizontal surface, if a mass is released and is located at coordinate x, the only force on the mass in the horizontal direction is the force of the spring, $\vec{F}_{spring} = -kx\hat{i}$. Newton's second law of motion, applied to the horizontal motion of m, then is $-kx\hat{i} = ma_x\hat{i}$. With a knowledge of calculus, the x-component of the position of the position vector with respect to time is $a_x = \frac{d^2x}{dt^2}$. Then we have $-kx = m\frac{d^2x}{dt^2}$. Now we we want to find a function of x(t) that satisfies this. From a knowledge of calculus we have $x(t) = A\cos(\omega t + \phi)$ (or $x(t) = A\sin(\omega t + \phi)$).



Taking the derivative of x(t), we find that the velocity function of Simple Harmonic Motion is $v(t) = -A\omega\sin(\omega t + \phi)$. The second derivative then gives us the acceleration function $a(t) = -A\omega^2\cos(\omega t + \phi)$.

These three equations allow us to determine the position, velocity, and acceleration of the mass at any time t.

Example:

A 0.5 kg mass is attached to a spring and completes one oscillation every 2 seconds. The range of oscillation is 0.4 m. given that the mass has zero speed when t=0 s, determine the position, velocity, and acceleration functions.

Solution:

We need $x(t) = A\cos(\omega t + \phi)$, so we know A=0.2 m, $\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2s} = 3.14\frac{rad}{s}$.

Now when t=0, v(t)=0 that is $v(t) = -A\omega\sin(\omega t + \phi) = 0$ or $\phi = 0$. Now we can write out the functions

$$x(t) = 0.2\cos(\pi t)$$

$$v(t) = (0.2)(-\pi)\sin(\pi t) = -0.2\pi\sin(\pi t)$$

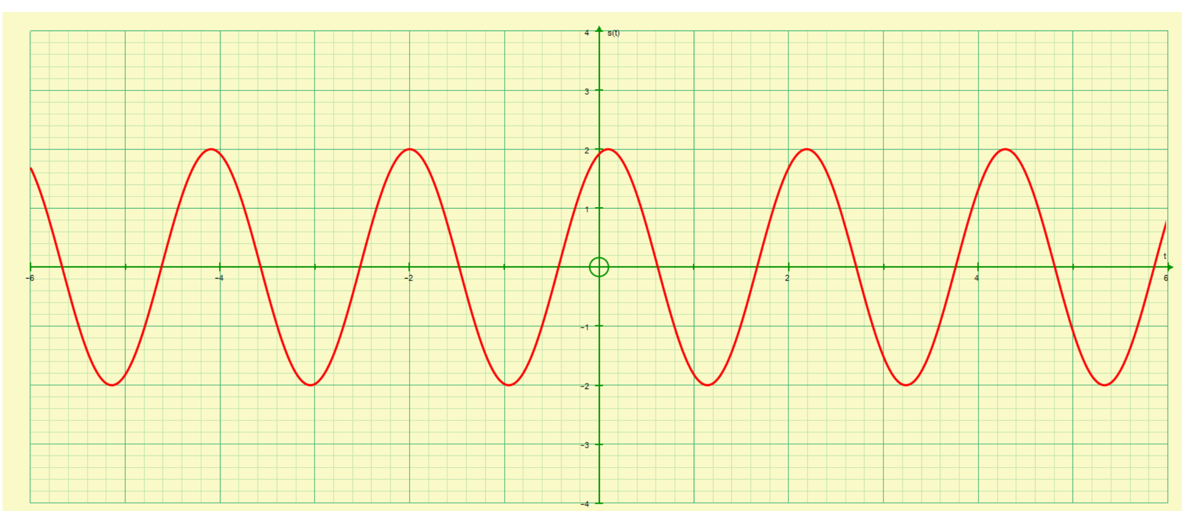
$$a(t) = (-0.2\pi)(\pi)\cos(\pi t) = -0.2\pi^2\cos(\pi t)$$

4.7 Simple Harmonic Motion

Physics Tool box

- **Force constant (k)** – the proportionality constant of a spring measured in newtons per metre.
- **Elastic Potential Energy (E_e)** – the energy stored in an object that is stretched, compressed, bent, or twisted.
- **Equilibrium Position** – the position of the mass when the spring is neither stretched nor compressed, since there is zero acceleration and zero total force on m at this location.
- The elastic potential energy stored in a spring is proportional to the force constant of the spring and to the square of the stretch or compression.
- **Simple Harmonic Motion (SHM)** – is the periodic vibratory motion such that the force (and thus the acceleration) is directly proportional to the displacement.
- A **reference circle** can be used to derive equations for the period and frequency of SHM.
- **Damped Harmonic Motion** – is periodic motion in which the amplitude of vibration and energy decrease with time.
- $T = 2\pi\sqrt{\frac{m}{k}}$ (T is period in seconds of one complete cycle)
- $f = \frac{1}{T}$ (frequency (measured in hertz (Hz)) is the number of cycles per second.

Any motion that repeats itself at regular intervals is called **periodic motion** or **harmonic motion**. What we are interested here is motion that repeats itself in a particular way – namely the displacement of the object from the origin is given as a function of time by the displacement function $s(t) = a\cos(\omega t + \phi)$ where a is the amplitude, ω is the angular frequency, and ϕ is called the phase angle.



This motion is called **simple harmonic motion (SHM)**. The displacement function is a sinusoidal function (even though represented by the cosine function).

The **period** (time for one complete cycle) is given by $T = 2\pi\sqrt{\frac{m}{k}}$ where m is the mass in kg and k is the force constant of the spring in N/m.

Since the **frequency** is the reciprocal of the period, $f = \frac{1}{T}$, then we have $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Example

A 0.54 kg mass is attached to a spring (which has a force constant of $1.5\times10^2\,\frac{N}{m}$). The mass spring is placed horizontally on a frictionless surface. The mass is displaced 20cm, and is then released. Determine the period and frequency of the Simple Harmonic Motion.

Solution:

$$m = 0.54kg,\ k = 1.5\times10^2\,\frac{N}{m},\ T = ?,\ f = ?$$

Using $T = 2\pi\sqrt{\frac{m}{k}}$

$$T = 2\pi\sqrt{\frac{0.54kg}{1.5\times10^2\,\frac{N}{m}}}$$
$$= 0.3769908s$$

The period for one cycle is 0.38s.

Now, $f = \frac{1}{T}$

$$f = \frac{1}{0.3769908s}$$
$$= 2.65258Hz$$

The frequency is 2.7 Hz.

Example

The maximum energy of a mass-spring system undergoing Simple Harmonic Motion is 5.64 J. The mass is 0.128kg and the force constant is 244 N/m.

a) What is the amplitude of the vibration?
b) Use two different approaches to determine the maximum speed of the mass
c) find the speed of the mass when it is 15.5 cm from the equilibrium position.

Solution:

a)

$$E_e = \frac{1}{2}kA^2$$

$$5.64J = \frac{1}{2}\left(244\frac{N}{m}\right)A^2$$

$$A^2 = \frac{2(5.64J)}{244\frac{N}{m}}$$

$$= 0.0462295\frac{mJ}{N}$$

$$= 0.0462295m\left(\frac{kg \cdot m^2}{s^2}\right)\left(\frac{s^2}{kg \cdot m}\right)$$

$$A = 0.21501m$$

The amplitude is 0.215 m

b)

Method 1:

$$E_k = \frac{1}{2}mv^2$$

$$v^2 = \frac{2E_k}{m}$$

$$v = \sqrt{\frac{2E_k}{m}}$$

$$= \sqrt{\frac{2(5.64J)}{0.128kg}}$$

$$= 9.38749\frac{m}{s}$$

The maximum speed is 9.39 m/s

Method 2:

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2$$
$$kx^2 = mv^2$$
$$v = \sqrt{\frac{kx^2}{m}}$$
$$= \sqrt{\frac{\left(244\frac{N}{m}\right)(0.21501m)^2}{0.128kg}}$$
$$= 9.387\frac{m}{s}$$

The maximum speed is 9.39 m/s

c)

$$E_{T1} = E_{T2}$$
$$\frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx_2^2$$
$$v^2 = \frac{kx_{\max}^2 - kx_2^2}{m}$$
$$v = \sqrt{\frac{k\left(x_{\max}^2 - x_2^2\right)}{m}}$$
$$= \sqrt{\frac{244\frac{N}{m}\left((0.215m)_{\max}^2 - (0.155m)_2^2\right)}{0.128kg}}$$
$$= 6.5052\frac{m}{s}$$

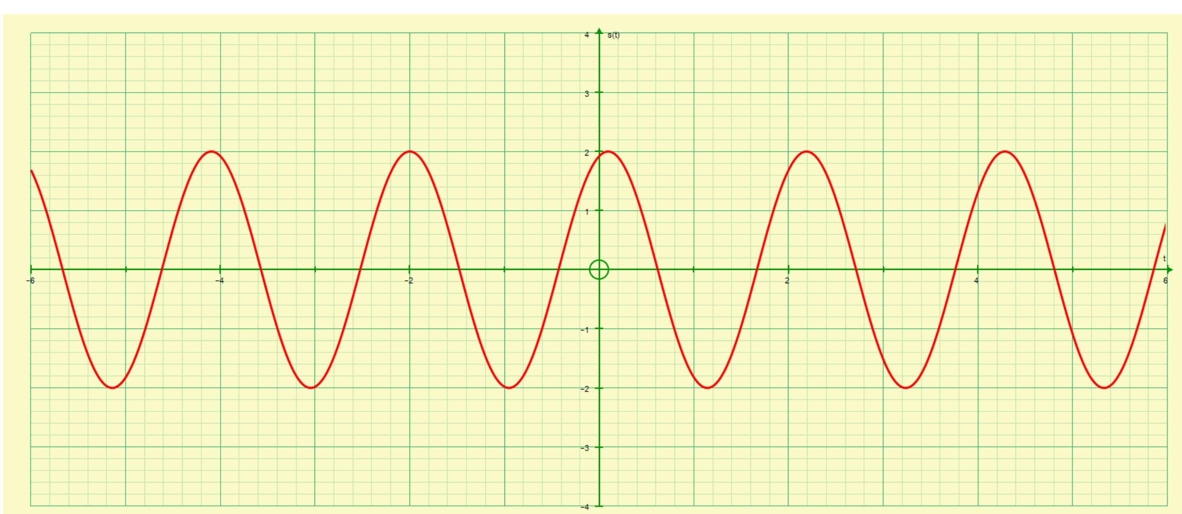
The speed is 6.51 m/s

4.7b Simple Harmonic Motion (Using Calculus)

Physics Tool box

- **Force constant (k)** – the proportionality constant of a spring measured in newtons per metre.
- **Elastic Potential Energy (E_e)** – the energy stored in an object that is stretched, compressed, bent, or twisted.
- **Equilibrium Position** – the position of the mass when the spring is neither stretched nor compressed, since there is zero acceleration and zero total force on m at this location.
- The elastic potential energy stored in a spring is proportional to the force constant of the spring and to the square of the stretch or compression.
- **Simple Harmonic Motion (SHM)** – is the periodic vibratory motion such that the force (and thus the acceleration) is directly proportional to the displacement.
- A **reference circle** can be used to derive equations for the period and frequency of SHM.
- **Damped Harmonic Motion** – is periodic motion in which the amplitude of vibration and energy decrease with time.
- Position function (t=0 is when stretched) $x(t) = A\cos(\omega t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$
- Velocity function $v(t) = -A\omega\sin(\omega t) = -A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)$
- Acceleration function $v(t) = -A\omega^2\cos(\omega t) = -A\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right)$

Any motion that repeats itself at regular intervals is called **periodic motion** or **harmonic motion**. What we are interested here is motion that repeats itself in a particular way – namely the displacement of the object from the origin is given as a function of time by the displacement function $s(t) = a\cos(\omega t + \phi)$ where a is the amplitude, ω is the angular frequency, and ϕ is called the phase angle.



This motion is called **simple harmonic motion (SHM)**. The displacement function is a sinusoidal function (even though represented by the cosine function).

The **period** (time for one complete cycle) is given by $T = 2\pi\sqrt{\frac{m}{k}}$ where m is the mass in kg and k is the force constant of the spring in N/m.

Since the **frequency** is the reciprocal of the period, $f = \frac{1}{T}$, then we have $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Example:

A block of mass 3.00 kg sits on a horizontal frictionless surface. It is attached to a spring with a spring constant of 36.0 N/m. The weight is pulled 0.16 m away from the equilibrium and released.

a) Assuming no damping, what is the speed of the block,
   i. as it passes through the equilibrium point?

The position function that describes the motion is $x = A\cos(\omega t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$

We use the cosine function because it is stretched at t=0.

Therefore the velocity function is $v = -A\omega\sin(\omega t) = -A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)$

We need only the time when the mass is at the equilibrium point (ie x=0). We can get that by setting the position function = to 0 and solving for time or by one quarter of the period

One quart of period: $T = 2\pi\sqrt{\frac{m}{k}} = \frac{\pi}{2}\sqrt{\frac{3}{36}}$

Setting position function equal to 0 and solving for t

$$0 = A\cos\left(\sqrt{\frac{36}{3}}t\right)$$

$$t = \sqrt{\frac{3}{36}}\cdot\frac{\pi}{2}$$

Plugging this time in into velocity function :

$$v = -(0.16)\sqrt{\frac{36}{3}}\sin\left(\sqrt{\frac{36}{3}}\left(\sqrt{\frac{3}{36}}\cdot\frac{\pi}{2}\right)\right) = -0.16\sqrt{\frac{36}{3}} = 0.554\frac{m}{s}$$

ii. when it has moved 5.0 cm from its release point? (*3 marks*)

Again setting position functions to (.16-.050) =0.11 m and solving for time

$$0.11 = 0.16\cos\left(\sqrt{\frac{36}{3}}\,t\right)$$

$$\cos\left(\sqrt{\frac{36}{3}}\,t\right) = 0.6875$$

$$\sqrt{\frac{36}{3}}\,t = \cos^{-1}(0.6875)$$

$$\sqrt{\frac{36}{3}}\,t = 0.812755$$

$$t = 0.2346$$

$$v = -(0.16)\sqrt{\frac{36}{3}}\sin\left(\sqrt{\frac{36}{3}}(0.2346)\right) = .40\frac{m}{s}$$

b) Find the time after release that the block first passed through the position x = –3.0 cm (on the opposite side of its equilibrium point). (*3 marks*)

$$x = A\cos(\omega t + \phi)$$

$$= A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$-0.030m = 0.160m\cos\left(\sqrt{\frac{36}{3}}t\right)$$

$$\cos\left(\sqrt{12}t\right) = -0.188$$

$$\sqrt{12}t = \cos^{-1}(-0.188)$$

$$t = 0.508s$$