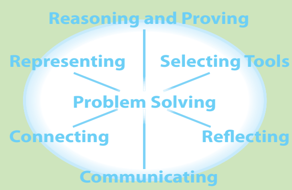
7 Exponents

Some relationships are linear, such as how the cost of bananas varies with the mass. Some relationships are quadratic, such as how the height of a space shuttle changes with time. Other relationships—such as how the intensity of sound varies with the distance from the source, and the growth and decay of organisms in nature—are exponential.

In this chapter, you will



- determine and describe the meaning of negative exponents and of zero as an exponent
- evaluate numerical expressions containing integer exponents and rational bases
- determine the exponent rules for multiplying and dividing numerical expressions involving exponents, and the exponent rule for simplifying numerical expressions involving a power of a power
- graph simple exponential relations, using paper and pencil, given their equations
- make and describe connections between representations of an exponential relation
- distinguish exponential relations from linear and quadratic relations by making comparisons
- collect data that can be modelled as an exponential relation from primary or secondary sources, and graph the data
- describe some characteristics of exponential relations arising from real-world applications by using tables of values and graphs
- pose and solve problems involving exponential relations arising from a variety of real-world applications by using graphs
- solve problems using given equations of exponential relations arising from a variety of real-world applications by substituting values for the exponent into the equations

Key Terms

doubling time
exponential decay
exponential growth
exponential relation
half-life

Ravi is a medical laboratory technologist. He prepares and analyses medical samples and administers various tests to patients, such as a test for the detection of cancer cells. He uses exponential equations to determine the growth rate of these cells. He trained for this position by taking a three-year course at St. Clair College.

Prerequisite Skills

Powers

1. Write each product as a power.

a) 6×6

b) $7 \times 7 \times 7 \times 7$

c) $(-2) \times (-2) \times (-2)$

d) $(4)(4)(4)(4)(4)(4)(4)(4)$

e) $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$

f) $\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right)$

2. Evaluate each power.

a) 5^2

b) 7^3

c) 10^5

d) $(-3)^2$

e) -3^2

f) -12^2

g) $\left(\frac{1}{2}\right)^2$

h) $\left(-\frac{1}{3}\right)^4$

i) $\left(-\frac{1}{5}\right)^3$

Linear Relations

3. Identify the slope and the y-intercept of each linear relation.

a) $y = 2x + 5$

b) $y = 3x - 1$

c) $y = -4x + 3$

d) $y = -\frac{1}{2}x - \frac{2}{3}$

4. Graph each linear relation. Label the y-intercept and any two other points.

a) $y = x - 3$

b) $y = 5x - 7$

c) $y = -2x + 6$

d) $y = \frac{1}{3}x + 2$

5. Ahmed earned $40 per day plus $2 per phone call as a marketing representative. Explain why this method of pay can be represented by a linear relation.

Evaluate Formulas

6. Substitute the indicated values. Evaluate for the remaining variable.

a) $A = \pi r^2$, $r = 5$ cm

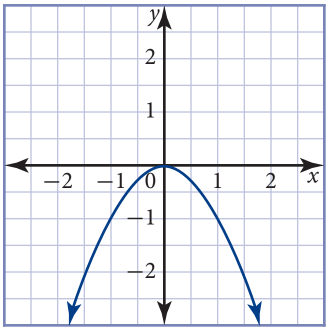
b) $I = Prt$, $P = \$200$, $r = 6\%$, $t = 2$ years

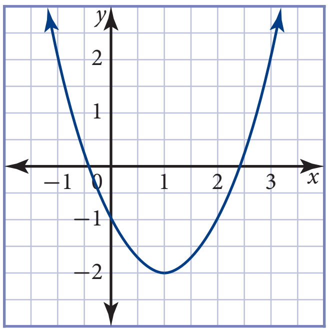
c) $V = s^3$, $s = 5$ m

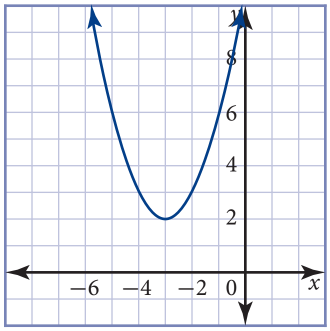
d) $P = 2(l + w)$, $l = 10$ cm, $w = 7$ cm

Quadratic Relations

7. Describe how each graph differs from the graph of $y = x^2$.

a) 

b) 

c) 

8. Graph each quadratic relation. Label the vertex and at least one other point on either side of the vertex.

a) $y = (x - 1)^2 + 4$

b) $y = 2(x + 3)^2 - 1$

c) $y = -x^2 + 1$

d) $y = -\frac{1}{2}(x - 4)^2 + 5$

Chapter Problem

Audiology technicians conduct hearing tests, which include tests for tone, speech reception, speech discrimination, and hearing threshold. Audiology technicians also prescribe and fit hearing aids and offer advice and counselling to patients. Educational requirements for audiology technicians include a 12- to 18-month college program.

Hearing loss can be a serious problem. In this chapter, you will investigate the effects of sound intensities on hearing and the consequences of extended exposure to loud sounds.

7.1 Exponent Rules

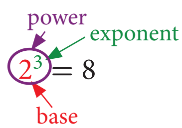
Very large numbers can be awkward to write. Sometimes, they are easier to work with when written in exponential form. For example, the intensity of an earthquake, the population of Earth, and the distance from Earth to the moon can be better expressed using exponents.

Investigate

Tools

- calculator

Patterns With Exponents



1. Multiply powers with the same base.

a) Copy and complete the table showing the expansion of each product.

Product	Expanded Form	Number of Factors	Single Power
$5^2 \times 5^4$	$(5 \times 5) \times (5 \times 5 \times 5 \times 5)$	6	5^6
$3^5 \times 3^2$			
$(-2)^5 \times (-2)^2$			
$(-3)^4 \times (-3)^3$			
$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)$			

b) **Reflect** Write a rule for the product of two powers with the same base.

2. Divide powers with the same base.

a) Copy and complete the table. Look for a pattern.

Quotient	Expanded Form	Number of Factors Remaining After Simplifying	Single Power
$\frac{5^6}{5^2}$	$\frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5}$	4	5^4
$\frac{3^5}{3^3}$			
$\frac{(-7)^3}{(-7)^2}$			
$4^7 \div 4^4$			
$\left(\frac{2}{3}\right)^4 \div \left(\frac{2}{3}\right)^3$			

b) **Reflect** Compare the single power to the original quotient. Write a rule for the quotient of two powers with the same base.

3. Find the power of a power.

a) Copy and complete the table. Look for a pattern.

Power of a Power	Expanded Form	Number of Factors of Given Base	Single Power
$(5^3)^2$	$(5 \times 5 \times 5)(5 \times 5 \times 5)$		
$(3^2)^4$			
$(2^2)^3$			
$(6^5)^2$			
$(4^3)^3$			

b) **Reflect** Compare the single power to the original power of a power. Write a rule for simplifying a power of a power.

4. Test your rules by simplifying each expression. Then use a calculator to check your rules.

a) $6^3 \times 6^2$

b) $5^2 \times 5^4$

c) $10^5 \div 10^4$

d) $7^6 \div 7^2$

e) $(10^4)^2$

f) $(5^2)^2$

Example 1

Simplify Expressions Involving Powers

Simplify.

a) $6^2 \times 6^3$ **b)** $\frac{7^5}{7^2}$ **c)** $(3^4)^3$ **d)** $\left(\frac{1}{2^3}\right)^2$

Solution

Method 1: Use the Exponent Rules

a) $$\begin{aligned} 6^2 \times 6^3 &= 6^{2+3} \\ &= 6^5 \\ &= 7776 \end{aligned}$$

b) $$\begin{aligned} \frac{7^5}{7^2} &= 7^{5-2} \\ &= 7^3 \\ &= 343 \end{aligned}$$

c) $$\begin{aligned} (3^4)^3 &= 3^{4 \times 3} \\ &= 3^{12} \\ &= 531\ 441 \end{aligned}$$

d) $$\begin{aligned} \left(\frac{1}{2^3}\right)^2 &= \frac{1^2}{2^{3 \times 2}} \\ &= \frac{1}{2^6} \\ &= \frac{1}{64} \end{aligned}$$

Method 2: Expand, Then Simplify

$$\begin{aligned} 6^2 \times 6^3 &= 36 \times 216 \\ &= 7776 \end{aligned}$$

$$\begin{aligned} \frac{7^5}{7^2} &= \frac{16\ 807}{49} \\ &= 343 \end{aligned}$$

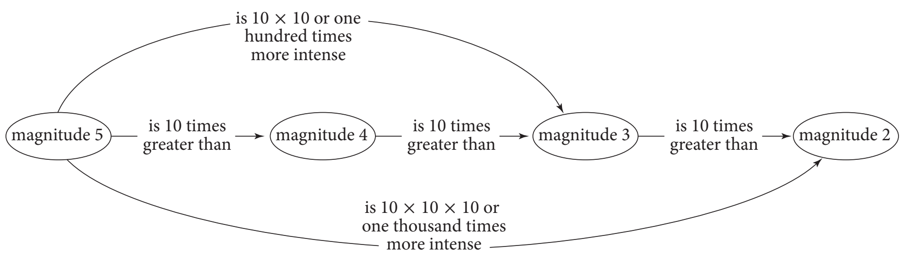
$$\begin{aligned} (3^4)^3 &= 81^3 \\ &= 531\ 441 \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2^3}\right)^2 &= \left(\frac{1}{8}\right)^2 \\ &= \frac{1}{64} \end{aligned}$$

Example 2

Earthquakes

The intensity of an earthquake can range from 1 to 10 000 000. The Richter scale is a base-10 exponential scale used to classify the magnitude of an earthquake. An earthquake with an intensity of 100 000 or 10^5, has a magnitude of 5 as measured on the Richter scale. The chart shows how magnitudes are related.



Intensity	Magnitude	Earthquake Effects
Up to $10^{2.5}$	2.5 or less	Usually not felt, but can be recorded by seismograph.
$10^{2.5}$ to $10^{5.4}$	2.5 to 5.4	Often felt, but only causes minor damage.
$10^{5.5}$ to $10^{6.0}$	5.5 to 6.0	Slight damage to buildings and other structures.
$10^{6.1}$ to $10^{6.9}$	6.1 to 6.9	May cause heavy damage in very populated areas.
$10^{7.0}$ to $10^{7.9}$	7.0 to 7.9	Major earthquake. Serious damage.
$10^{8.0}$ and greater	8.0 or greater	Great earthquake. Can totally destroy communities near the epicentre.

An earthquake measuring 2 on the Richter scale can barely be felt, but one measuring 6 often causes damage. An earthquake with magnitude 7 is considered a major earthquake.

a) How much more intense is an earthquake with magnitude 6 than one with magnitude 2?

b) How much more intense is an earthquake with magnitude 7 than one with magnitude 6?

Math Connect

The Apollo astronauts discovered disturbances on the moon, now known as moonquakes. They are much weaker than earthquakes. Most are caused by tidal effects from the Earth and the sun. For more information on earthquakes and moonquakes, go to *www.mcgrawhill.ca/links/foundations11* and follow the links.

Solution

Earthquakes are compared by dividing their intensities.

a) An earthquake with magnitude 6 has an intensity of 10^6 and one with magnitude 2 has an intensity of 10^2.

$$\begin{aligned}\frac{10^6}{10^2} &= 10^{6-2}\\ &= 10^4\\ &= 10\,000\end{aligned}$$

An earthquake with magnitude 6 is 10 000 times as intense as one with magnitude 2.

b)
$$\begin{aligned}\frac{10^7}{10^6} &= 10^{7-6}\\ &= 10^1\\ &= 10\end{aligned}$$

An earthquake with magnitude 7 is 10 times as intense as one with magnitude 6.

Key Concepts

- To multiply powers with the same base, keep the base the same and add the exponents.
  $a^p \times a^q = a^{p+q}$
  For example, $3^5 \times 3^3 = 3^8$.
- To divide powers with the same base, keep the base the same and subtract the exponents.
  $a^p \div a^q = a^{p-q}$
  For example, $5^7 \div 5^3 = 5^4$.
- To simplify a power of a power, keep the base the same and multiply the exponents.
  $(a^p)^q = a^{p \times q}$
  For example, $(2^3)^4 = 2^{12}$.

Discuss the Concepts

D1. The magnitudes of most earthquakes are between 0 and 10 on the Richter scale. These magnitudes correspond to intensities of between 1 and 10 000 000 000. Explain how the magnitude of an earthquake, as measured by the Richter scale, is related to its intensity.

D2. Evaluate $\left(\frac{1}{81}\right)^3$ and $\left(\frac{1}{9}\right)^6$. Use the exponent rules to explain why the answers are the same.

D3. Maggie evaluated the following problem. Her solution is shown.

$2^3 \times 2^2 = 2^6$

Is her solution correct? If not, explain where she went wrong and correct her work.

Practise A

For help with questions 1 to 5, refer to Example 1.

1. Write each expression as a single power, then evaluate.

a) $5^2 \times 5^2$ **b)** $2^4 \times 2^3$ **c)** $(-3)^2 \times (-3)^4$

d) $(-4)^3 \times (-4)^3$ **e)** $\left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^3$ **f)** $\left(-\frac{1}{2}\right)^2 \times \left(-\frac{1}{2}\right)$

2. Write each expression as a single power, then evaluate.

a) $6^5 \div 6^4$ **b)** $8^7 \div 8^5$ **c)** $12^8 \div 12^7$

d) $\frac{2^{10}}{2^6}$ **e)** $\frac{(-2)^9}{(-2)^6}$ **f)** $\frac{(-3)^6}{(-3)^4}$

3. Write the single powers, then evaluate.

a) $(5^2)^3$ **b)** $(2^3)^3$

c) $[(-4)^3]^2$ **d)** $\left(\frac{1}{7^2}\right)^2$

e) $\left(\frac{1}{3^3}\right)^2$ **f)** $\left(-\frac{1}{10^2}\right)^4$

4. Show two ways of evaluating each expression.

a) $6^2 \times 6^3$ **b)** $7^4 \times 7^2$

c) $9^5 \div 9^3$ **d)** $\frac{(-7)^4}{(-7)^3}$

e) $(5^2)^3$ **f)** $(10^5)^2$

g) $(-8)^3(-8)$ **h)** $[(-1)^{11}]^9$

5. Write each expression as a single power, then evaluate.

a) $9^4 \times 9^5$ **b)** $(7^2)^4$

c) $(-6) \times (-6)^5$ **d)** $24^6 \div 24^5$

e) $\frac{9^7}{9^5}$ **f)** $\left(\frac{3}{4}\right)^5 \times \left(\frac{3}{4}\right)^2$

g) $(4^3)^5$ **h)** $\frac{(-8)^9}{(-8)^6}$

i) $\left(-\frac{5}{7}\right)^8 \div \left(-\frac{5}{7}\right)^4$

For help with questions 6–8, refer to Example 2.

6. Canada's greatest earthquake was recorded in 1949 at the Queen Charlotte Islands in British Columbia. It had a magnitude of about 8. The magnitude of the greatest recorded earthquake in Ontario was about 6. It occurred at the Ontario—Quebec border north of Mattawa in 1935. How much more intense was the earthquake in British Columbia compared to the one in Ontario?

7. The earthquake in the Indian Ocean that caused the devastating tsunami in 2004 measured 9. An earthquake measuring 4 occurred off Vancouver Island in June 2006. How much more intense was the earthquake in the Indian Ocean compared to the one off Vancouver Island?

8. On December 6, 2006 there was an earthquake with magnitude 4.2 near Cochrane, ON. Another earthquake, with magnitude 2.8 occurred on January 29, 2007 near Hawkesbury, ON. In each case, local residents reported feeling the earthquake. Which earthquake was more intense? How much more?

Apply B

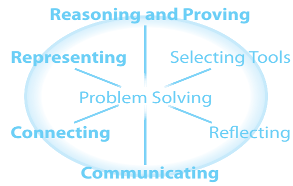
Literacy Connect

9. You can write 3^8 as $3^2 \times 3^6$ using the exponent rules.

a) Write 3^8 as the product of two powers in three other ways.

b) Write 2^5 as the quotient of two powers in three ways.

c) Write 7^{12} as a power of a power in three ways.



10. Consider the powers 64^2 and 16^3.

a) Are these powers equivalent? Use the exponent rules to explain.

b) Write a power with a different base that is equivalent to 64^2.

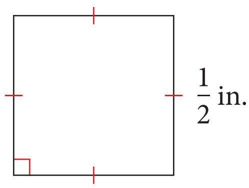
11. The probability of rolling a 5 using a single die is $\frac{1}{6}$. The probability of rolling two 5s using two dice is $\left(\frac{1}{6}\right) \times \left(\frac{1}{6}\right)$ or $\left(\frac{1}{6}\right)^2$.

a) Evaluate the probability of rolling two 5s. Leave your answer in fraction form.

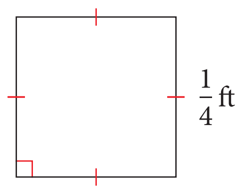
b) What would be the probability of rolling three 5s using three dice?

12. The area of a square can be calculated using the formula $A = s^2$, where s is the length of a side. Calculate the area of a square with each side length. Express your answer as a fraction.

a)



b)



13. There are 12 in. in 1 ft.

a) Convert your answer in question 12a) to square feet.

b) Convert your answer in question 12b) to square inches.

14. Copy and complete the table.

Measurement to be Calculated	Formula	Dimensions Given	Calculated Measurement
Area of a Circle	$A = \pi r^2$	$r = \pi$ cm	
Volume of a Cube	$V = s^3$	$s = \frac{1}{2}$ in.	
Volume of a Sphere	$V = \frac{4}{3}\pi r^3$	$r = \frac{1}{8}$ in.	
Volume of a Cylinder	$V = \pi r^2 h$	$r = h = 5$ cm	

15. Rubik's Cube® is a large cube made of small congruent cubes. Each small cube has edges about 2 cm long. The cubes on each face of the Rubik's Cube® are arranged in 3 rows of 3. What is the approximate volume of the Rubik's Cube®?

Extend C

16. Use the exponent rules to simplify each expression. Then use a calculator to evaluate.

a) $10^{3.1} \times 10^{4.2}$

b) $\frac{10^{7.9}}{10^{3.1}}$

c) $2^{4.8} \times 2^{1.6}$

d) $\left(\frac{1}{2}\right)^{7.8}\left(\frac{1}{2}\right)^{1.1}$

17. Use the exponent rules to simplify each expression.

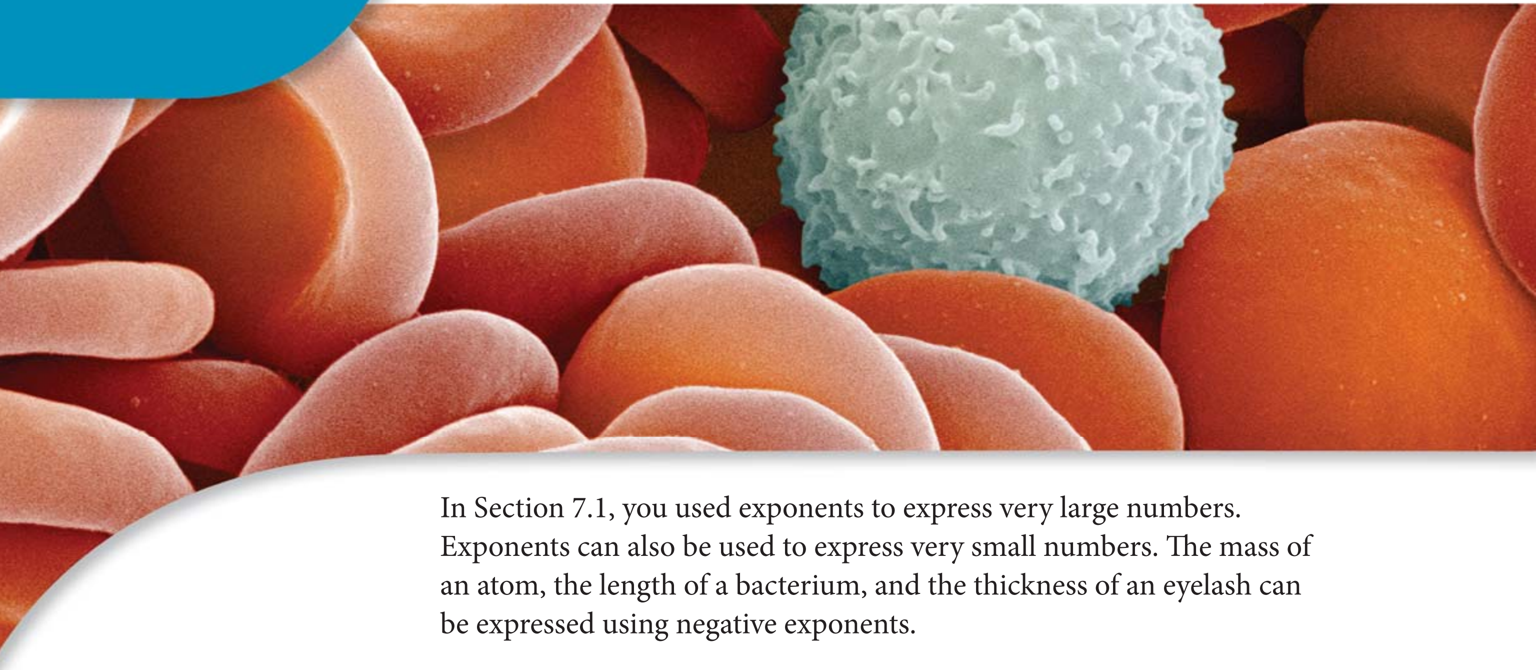
a) $(4x^3)(2x^4)$

b) $\frac{-12a^5b^3}{3a^2b}$

c) $(m^2n^3)^5$

d) $\left(\frac{k^5h^2}{k^2}\right)^3$

7.2 Zero and Negative Exponents



In Section 7.1, you used exponents to express very large numbers. Exponents can also be used to express very small numbers. The mass of an atom, the length of a bacterium, and the thickness of an eyelash can be expressed using negative exponents.

Investigate

The Meaning of Zero and Negative Exponents

Use patterns to evaluate powers with zero or negative exponents.

1. Copy and complete the statements. Describe each pattern.

a) $2^5 = 32$
$2^4 = 16$
$2^3 = \blacksquare$
$2^2 = \blacksquare$
$2^1 = \blacksquare$
$2^0 = \blacksquare$

b) $3^5 = 243$
$3^4 = \blacksquare$
$3^3 = \blacksquare$
$3^2 = \blacksquare$
$3^1 = \blacksquare$
$3^0 = \blacksquare$

c) $10^5 = 100\ 000$
$10^4 = \blacksquare$
$10^3 = \blacksquare$
$10^2 = \blacksquare$
$10^1 = \blacksquare$
$10^0 = \blacksquare$

2. Reflect For each set of powers, what is true about the values of the powers with zero exponents? Express this as a general rule.

3. Continue your patterns from question 1 to evaluate the powers with negative exponents. Express your answers as whole numbers or fractions.

a) $2^3 = \blacksquare$
$2^2 = \blacksquare$
$2^1 = \blacksquare$
$2^0 = \blacksquare$
$2^{-1} = \blacksquare$
$2^{-2} = \blacksquare$

b) $3^3 = \blacksquare$
$3^2 = \blacksquare$
$3^1 = \blacksquare$
$3^0 = \blacksquare$
$3^{-1} = \blacksquare$
$3^{-2} = \blacksquare$

c) $10^3 = \blacksquare$
$10^2 = \blacksquare$
$10^1 = \blacksquare$
$10^0 = \blacksquare$
$10^{-1} = \blacksquare$
$10^{-2} = \blacksquare$

4. Reflect Explain how to evaluate a power with a negative exponent.

Example 1

Evaluate Powers With Zero or Negative Exponents

Evaluate.

a) 8^{-2}

b) 7^0

c) $(-4)^{-1}$

d) 3^{-2}

Solution

a) $8^{-2} = \frac{1}{8^2}$

$= \frac{1}{64}$

b) $7^0 = 1$

c) $(-4)^{-1} = \frac{1}{(-4)^1}$

$= -\frac{1}{4}$

$\frac{1}{-4} = -\frac{1}{4}$

d) $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$

Example 2

Simplify Expressions

The rules for positive exponents also work for zero and negative exponents. Use the exponent rules to evaluate.

a) $4^3 \times 4^{-5}$

b) $\frac{(-2)^2}{(-2)^{-5}}$

c) $\left(\frac{4^2}{4^5}\right)^2$

Solution

a) $4^3 \times 4^{-5} = 4^{3+(-5)}$

$= 4^{-2}$

$= \frac{1}{4^2}$

$= \frac{1}{16}$

b) $\frac{(-2)^2}{(-2)^{-5}} = (-2)^{2-(-5)}$

$= (-2)^7$

$= -128$

c) $\left(\frac{4^2}{4^5}\right)^2 = (4^{-3})^2$

$= 4^{-6}$

$= \frac{1}{4^6}$

$= \frac{1}{4096}$

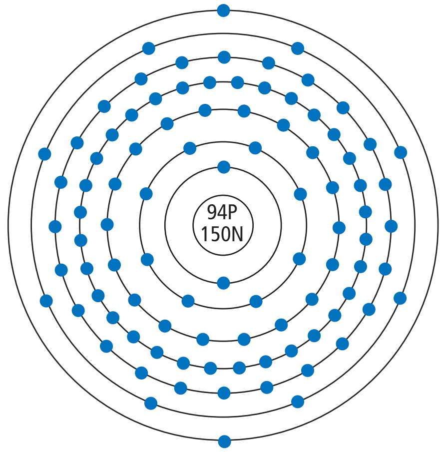
Example 3

Radioactive Decay

Mr. Roberts presented his math class with the following problem involving negative exponents. Every 80 million years, 2^{-1} of the mass of a sample of plutonium-244 decays to a different element. If the original mass of a sample of plutonium-244 was 16 g, determine the mass remaining after

a) 80 million years

b) 240 million years



Plutonium-244

Solution

a) $$\begin{aligned}\text{Amount remaining} &= 16 \times 2^{-1} \\ &= 16 \times \frac{1}{2} \\ &= 8\end{aligned}$$

After 80 million years, 8 g of plutonium-244 would remain.

b) $$\begin{aligned}\text{Number of time units for decay} &= 240 \text{ million} \div 80 \text{ million} \\ &= 3\end{aligned}$$

Since the decay occurs over 3 units of time,

$$\begin{aligned}\text{the amount remaining} &= 16 \times (2^{-1})^3 \\ &= 16 \times (2^{-3}) \\ &= 16 \times \frac{1}{2^3} \\ &= 16 \times \frac{1}{8} \\ &= 2\end{aligned}$$

After 240 million years, 2 g of plutonium-244 would remain.

Key Concepts

- Any base raised to an exponent of zero equals 1.
  $x^0 = 1$
  For example, $6^0 = 1$.
- Any base raised to a negative exponent is equal to the reciprocal of the base raised to a positive exponent.
  $x^{-a} = \frac{1}{x^a}$ $\quad\quad$ $\frac{1}{x^{-a}} = x^a$
  For example,
  $$7^{-2} = \frac{1}{7^2} = \frac{1}{49} \quad\quad \frac{1}{2^{-3}} = 2^3 = 8$$

Discuss the Concepts

D1. Describe the steps you would use to evaluate each power.

a) 2^{-3}

b) $\frac{1}{3^{-4}}$

D2. Evaluate.

a) $(-2)^4$ **b)** -2^4 **c)** 2^{-4}

Explain how the powers are different. Draw and label a number line to help you explain.

D3. Refer to the rules for working with positive exponents. Copy and complete the table. Use examples to show your understanding of the exponent rules involving negative exponents.

Product	Expanded Form	Number of Factors	Single Power

Practise A

For help with question 1, refer to the Investigate.

1. a) Write $\frac{1}{9^5}$ as a power with base 9.

b) Write 6^3 as a power with base $\frac{1}{6}$.

c) Write 5^{-2} as a power with base $\frac{1}{5}$.

d) Write $\frac{1}{4^{-1}}$ as a power with base 4.

For help with questions 2 to 5, refer to Example 1.

2. Evaluate. Express your answers as whole numbers or fractions.

a) $5^2, 5^{-2}$ **b)** $2^1, 2^{-1}$

c) $4^4, 4^{-4}$ **d)** $10^3, 10^{-3}$

e) $1^6, 1^{-6}$ **f)** $2^9, 2^{-9}$

g) $(-3)^4, (-3)^{-4}$ **h)** $(-8)^1, (-8)^{-1}$

3. Evaluate. Express your answers as whole numbers or fractions.

a) 12^0 **b)** 8^{-1}

c) 6^{-2} **d)** $100\ 000^0$

e) 500^{-1} **f)** 5^{-3}

g) $(-2)^{-8}$ **h)** $(-10)^{-3}$

i) $\left(\frac{1}{6}\right)^{-2}$ **j)** 3^{-5}

k) $\left(\frac{1}{3}\right)^{-3}$ **l)** $(-7)^3$

4. Use a pattern to show that $4^{-3} = \frac{1}{4^3}$.

5. Evaluate each of the powers in question 3 using a calculator. Explain the benefits of each form.

6. Simplify each quotient two ways.

- Write the powers in expanded form and eliminate factors common to the numerator and the denominator.
- Use the exponent rules.

Compare the results. How can you use the results from both methods to explain the meaning of a negative exponent?

a) $\frac{8^7}{8^5}$ **b)** $\frac{5^4}{5^9}$ **c)** $\frac{7}{7^3}$

d) $\frac{12^5}{12^8}$ **e)** $\frac{(-4)^7}{(-4)^8}$ **f)** $\frac{(-3)^2}{(-3)^7}$

7. Simplify each quotient two ways.

- Write the powers in expanded form and eliminate factors common to the numerator and the denominator.
- Use the exponent rules.

Compare the results. How can you use the results from both methods to explain the meaning of a zero exponent?

a) $\frac{6^5}{6^5}$ **b)** $\frac{8^4}{8^4}$ **c)** $\frac{16^6}{16^6}$

d) $\frac{2^7}{2^7}$ **e)** $\frac{(-9)^3}{(-9)^3}$ **f)** $\frac{(-7)^2}{(-7)^2}$

For help with question 8, refer to Example 2.

8. Rewrite each as a single power, then evaluate. Express your answers as fractions.

a) $8^3 \times 8^{-1}$

b) $\frac{4^2}{4^{-1}}$

c) $\frac{1}{(2^4)^3}$

d) $(-3)^3(-3)^{-1}$

e) $(10^{-2})^3$

f) $\left(\frac{1}{2^4}\right)\left(\frac{1}{2^4}\right)$

g) $6^2 \div 6^5$

h) $5^{-7} \times 5^4$

i) $(4^3)^{-2}$

j) $\left(\frac{1}{3}\right)^{-6} \times \left(\frac{1}{3}\right)^3$

k) $\left(\frac{1}{9}\right)^{-9} \times \left(\frac{1}{9}\right)^7$

l) $(5^{-2})^3$

For help with question 9, refer to Example 3.

9. A second question on Mr. Roberts' math test is shown.

Radium-226 is a radioactive element that decays by 2^{-1} of its mass after about 1600 years. Determine the remaining mass of 16 g of radium-226 after

a) 1600 years

b) 8000 years

Apply B

Literacy Connect

Math Connect

One million seconds is equal to 11.5 days. One billion seconds is equal to 31.7 years!

10. How small is one billionth?

a) Write one thousand as a power of 10.

b) Write one thousandth as a fraction, as a power of $\frac{1}{10}$, and then as a power of 10.

c) Write one millionth as a power of 10.

d) Write one billionth as a power of 10.

e) Write the ratio of one thousandth to one billionth as a power of 10. How many times larger is one thousandth compared to one billionth?

11. Evaluate the power in each statement. Express your answers as whole numbers or fractions.

a) One kilobyte is 2^{10} bytes.

b) One byte is 2^{-10} kilobytes.

c) One megabyte is $(2^{10})^2$ bytes.

d) One byte is $(2^{-10})^3$ gigabytes.

e) One bit is 2^{-3} bytes.

f) One bit is $2^{-40} \times 2^{-3}$ terabytes.

Chapter Problem

12. Sound intensity levels are recorded in decibels (dB). The actual intensity is recorded in Watts per square metre (W/m^2). The faintest sound that can be heard by the human ear has an intensity of 10^{-12} W/m^2 and is assigned an intensity level of 0 dB. A sound that is 10 times more intense is assigned a sound level of 10 dB. A sound that is 10×10 or 100 times more intense is assigned a sound level of 20 dB, and so on. To calculate how much more intense sound A is than sound B, divide their intensity levels by 10 to get a and b, then use the ratio $\frac{10^a}{10^b}$.

Source	Intensity Level
Threshold of hearing	0 dB
Rustling leaves	10 dB
Whisper	20 dB
Normal conversation	60 dB
Busy street traffic	70 dB
Vacuum cleaner	80 dB
MP3 player at maximum level	100 dB
Front row of seating at a rock concert	110 dB
Threshold of pain	130 dB
Military jet takeoff	140 dB
Instant perforation of eardrum	160 dB

a) How much more intense is the sound from normal conversation compared to the threshold of hearing?

b) How much more intense is sound from the front row of a rock concert compared to the sound from busy street traffic?

c) Some dogs have a threshold of hearing of −5 dB. How much less intense is a dog's threshold of hearing compared to a human's threshold of hearing?

Extend C

13. Loudness describes the strength of the ear's perception of a sound. The exponent must be increased by a factor of 10 for a sound to sound twice as loud.

a) Refer to the ratio in question 12. What is the corresponding increase in loudness?

b) Write this rule using powers.

c) How many vacuum cleaners would it take to sound twice as loud as one vacuum cleaner? Explain.

14. To estimate how much an item costing T dollars in 2007 would have cost in a given year (after 1914), C, you can use the formula $C = T(1.0323)^{-n}$, where n is the number of years before 2007.

a) How much would a \$150 coat have cost in 1920?
Hint: Substitute the number of years before 2007 for n.

b) How much would a \$20 000 car have cost in 1970?

c) How much would a \$1.99 bag of dried fruit snacks have cost in 1962?

d) How much would a \$200 000 condominium have cost in 1980?

15. An orange peel may take 1 year to decay to 10^{-1} of its original mass. An aluminum can may take 50 years to decay to 10^{-1} of its original mass.

a) An orange peel has an original mass of 5 g. What mass will remain after 2 years?

b) A metal can has an original mass of 16.5 g. What mass will remain after 100 years?

16. Write each numerator and denominator as a power, then use the exponent rules to simplify. Express your answer as a power with a whole number base.

a) $\frac{128}{1024}$

b) $\frac{243}{6561}$

c) $\frac{3125}{625}$

d) $\frac{49}{2401}$

e) $\frac{11}{1331}$

f) $\frac{512}{64}$

g) $\frac{1}{8} \times \frac{1}{16}$

h) $\frac{1}{25} \times \frac{1}{125}$

17. Suppose $y = 4^2$ and $z = 4^3$. Write each expression as a power with base 4.

a) $y^{-1}z^2$

b) $\frac{y}{z}$

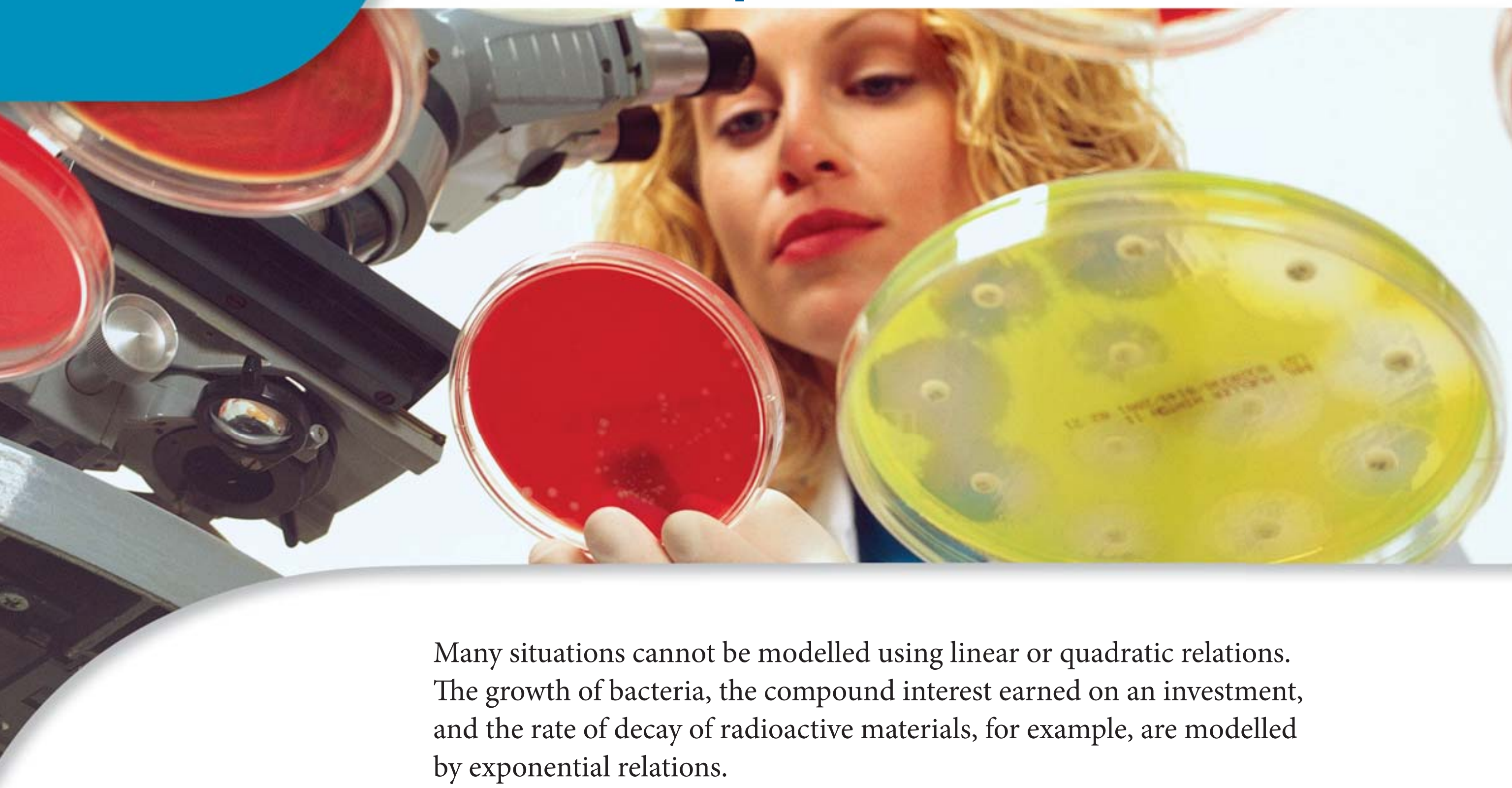
c) $\frac{z^{-4}}{y}$

d) y^5z^{-2}

e) $\frac{y^{-2}z^{-4}}{y}$

f) $\frac{yz}{y^2z^{-1}}$

7.3 Investigate Exponential Relationships



Many situations cannot be modelled using linear or quadratic relations. The growth of bacteria, the compound interest earned on an investment, and the rate of decay of radioactive materials, for example, are modelled by exponential relations.

Investigate 1

Tools

- 3 sheets of paper, letter-sized or bigger

Paper Folding

1. a) Copy the table. Add enough rows for seven folds or stages.

Stage	Total Number of Rectangles

b) Take a sheet of paper. Fold the paper about 1 cm in from one edge, as shown. You will create a rectangle that is 1 cm by about 28 cm if using a 21.5 cm by 27.8 cm sheet of paper. Record the number of rectangles for stage 1. Do not count the larger rectangle that is left over.

c) Fold the paper again 2 cm from the edge. Record the number of rectangles when there are two folds. Continue folding in a fan-fold manner, for 3 cm from the edge, 4 cm, 5 cm, and so on, as shown. Complete the table.

d) Make a scatter plot of the data.

e) Describe the shape of the graph.

2. a) Copy the table. Add enough rows for 5 folds or stages.

Stage	Total Number of Rectangles

b) Take another sheet of paper. Fold the paper about 1 cm in from one edge. Record the number of rectangles for stage 1.

c) Fold the paper as in step 1c), except fold the paper twice while still keeping a 2 cm edge. Record the number of rectangles. Then fold the paper three times keeping a 3 cm edge, four times keeping a 4 cm edge, and so on. Complete the table.

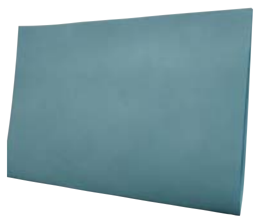
d) Make a scatter plot of the data.

e) Describe the shape of the graph.

3. a) Copy the table. Add enough rows for seven folds or stages.

Stage	Total Number of Rectangles

b) Take a third sheet of paper. Fold it in half across the length of the paper to make two layers, as shown. Record the number of rectangles created.



c) Fold it in half again to make four layers. Record the number of rectangles. Continue until you have made seven folds. Complete the table.

d) Make a scatter plot of the data.

e) Describe the shape of the graph.

4. **Reflect** In question 1, the data can be represented by a linear relation. In question 2, the data can be represented by a quadratic relation and in question 3, the data can be represented by an **exponential relation**.How are the graphs of these relations similar? different?

exponential relation

- a relation that can be represented by the form $y = a^x$, where a is a positive constant and $a \neq 1$.
- the ratios of consecutive y-values are constant

Investigate 2 Newton's Law of Cooling

Tools

- temperature probe
- hot water
- large bowl

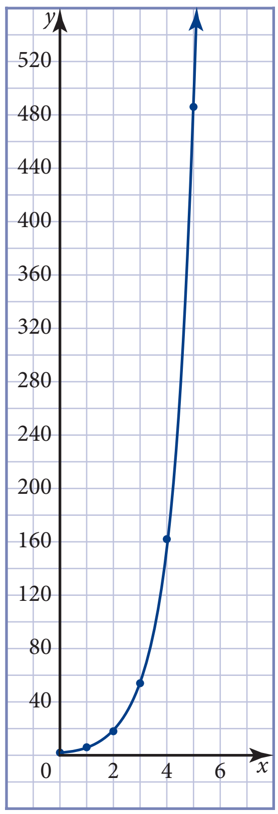
Optional:

- thermometer

1. Set up a temperature probe to collect temperature data every 5 min.
2. Place the temperature probe in the bowl of hot water and turn it on immediately. (Alternatively, use a regular thermometer and record the temperature every 5 min.)
3. After 60 min, turn off the probe.
4. Make a scatter plot of the temperature versus time, with time on the horizontal axis.
5. Describe the shape of the graph.
6. Divide each temperature by the previous one in the chart (e.g., divide the temperature after 10 min by the temperature after 5 min).
7. **Reflect** How do the results compare?

In an exponential relation, for equal steps of x, the ratios of consecutive y-values are constant. In the table, dividing each value y by the previous value of y gives a result of 3, so as x increases by 1, y increases by a factor of 3. Because each ratio is constant, the relationship is exponential.

x	y	Ratio of Successive y-values
0	2	
1	6	$6 \div 2 = 3$
2	18	$18 \div 6 = 3$
3	54	$54 \div 18 = 3$
4	162	$162 \div 54 = 3$
5	486	$486 \div 162 = 3$



The graph increases rapidly as you move to the right on the x-axis, and approaches a vertical line.

Example

Pendulum Motion

A large pendulum was set in motion. With each complete swing, the pendulum's maximum distance from its rest position decreased. A motion sensor was used to obtain the data after every 5 swings.

Number of Swings	Maximum Distance (cm)
0	23.5
5	19.9
10	17.0
15	14.6
20	10.4
25	8.7
30	7.6

a) Make a scatter plot of the data. Describe the graph.

b) Calculate the ratio between successive distances. Is the relationship between the number of swings and the maximum distance of the pendulum swing exponential? Explain.

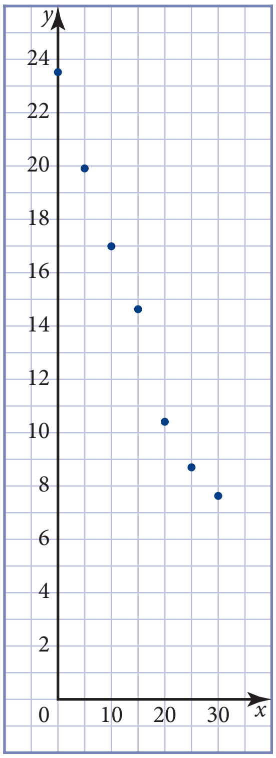
c) Is this relationship an example of **exponential growth** or **exponential decay**?

exponential growth
- non-linear growth represented by an exponential relation and a graph with an upward curve

exponential decay
- non-linear growth represented by an exponential relation and a graph with a downward curve

Solution

a)



The initial distance was 23.5 cm. The maximum distance declines rapidly at the beginning, but levels off as the number of swings increases. Eventually, the graph will approach a horizontal line.

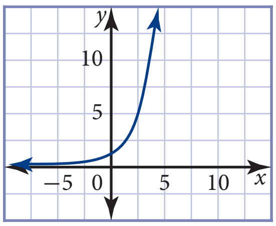
b)

Number of Swings	Maximum Distance (cm)	Ratio of Successive Distances
0	23.5	
5	19.9	19.9 ÷ 23.5 = 0.847
10	17.0	0.853
15	14.6	0.861
20	10.4	0.712
25	8.7	0.836
30	7.6	0.877

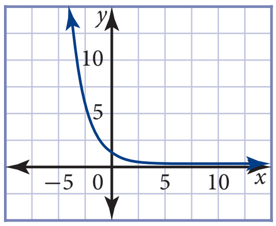
c) The ratio between successive distances is approximately 0.85, indicating a relatively constant rate of change. (The differences in the ratios might be due to measurement error.) Therefore, the relationship between the number of swings and the swing distance is an example of exponential decay.

Key Concepts

- The graph of an exponential relation is a curve that is approximately horizontal at one end and increases or decreases rapidly at the other end.



Exponential Growth



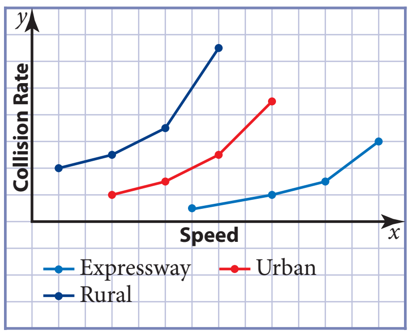
Exponential Decay

- For any exponential relation, the ratio between successive terms is constant.

Discuss the Concepts

A scientific study showed a relationship between the speed of a car and the chance of a collision.

Collision Rates on Various Road Types



D1. Is the relationship between speed and the chance of a collision exponential? Justify your answer.

D2. On which road type is the chance of a collision the greatest? Suggest reasons why this might be true.

Practise A

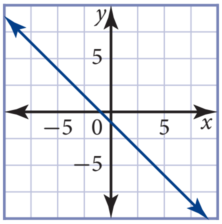
For help with questions 1 to 3, refer to the Example.

1. Show that this relation is exponential.

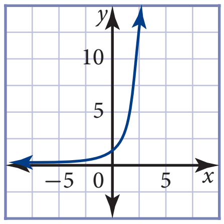
x	y
1	2
2	4
3	8
4	16
5	32
6	64

2. Which of these graphs could represent an exponential relation? Explain.

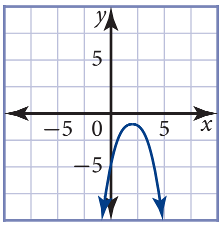
a)



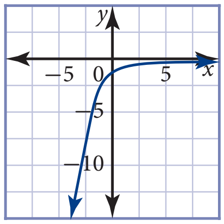
b)



c)



d)



3. In an old story, *The King's Chessboard*, a king rewards a farmer for saving the kingdom. The king gives the farmer 1 grain of rice on the first day, 2 grains of rice on the second day, 4 on the third day, and so on, doubling the number of grains of rice each day, for 64 days.

a) Make a table showing the number of grains of rice the farmer receives each day for the first 7 days.

b) How does this story illustrate exponential growth?

c) How many grains of rice will the farmer receive on the sixteenth day?

d) No matter how rich the king is, he will eventually run out of rice. Explain why.

Apply B

4. Current radio dials are digital. Look at an older analogue AM or FM radio dial and you will see that the frequencies are not spaced evenly.

   a) Copy and complete the table. Measure the distance, accurate to the nearest tenth of a centimetre, from the left end of the radio dial to each radio frequency on the dial.

Distance (cm)	AM Radio Frequency (kHz)
	540
	600
	700
	900
	1200
	1400
	1600

   b) Make a scatter plot of the data. Let distance be the independent variable.

   c) Is the relationship between the distance and frequency exponential? Justify your answer.

5. On a television game show, the cash prizes were designed to resemble exponential growth.

   The prizes are:

\$100	\$2000	\$64 000
\$200	\$4000	\$125 000
\$300	\$8000	\$250 000
\$500	\$16 000	\$500 000
\$1000	\$32 000	\$1 000 000

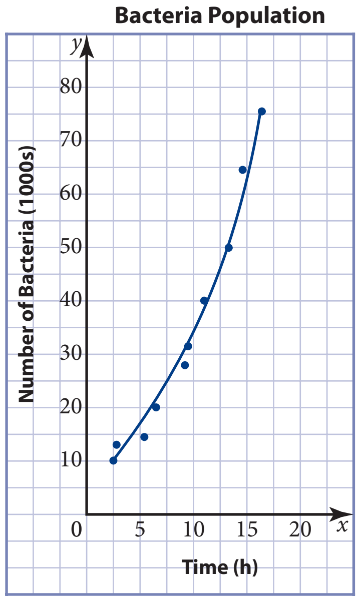
   a) Show that these cash prizes do not actually grow exponentially.

   b) Make a new table of 15 cash prizes that do grow exponentially.

   c) Find other examples of cash prizes that seem to—but do not—grow exponentially.

6. The table and graph show the number of bacteria in 1 cm^3 of a bacterial culture over a period of hours.

Time (h)	Number of Bacteria (1000s)
2.5	10.07
2.8	13.07
5.4	14.59
6.5	20.70
9.2	27.94
9.5	31.50
11.0	40.04
13.3	49.90
14.6	64.72
16.4	75.57



a) Describe the shape of the graph.

b) Estimate the number of bacteria present at the beginning of the test. Explain how you got your answer.

c) Estimate the number of bacteria present after 10 h.

d) What is the trend in the bacteria growth?

7. Bacteria tend to grow exponentially, by a common factor over equal time intervals, because each cell divides into two daughter cells. A particular bacteria culture begins with 1000 bacteria. The number of bacteria doubles about every 12 h.

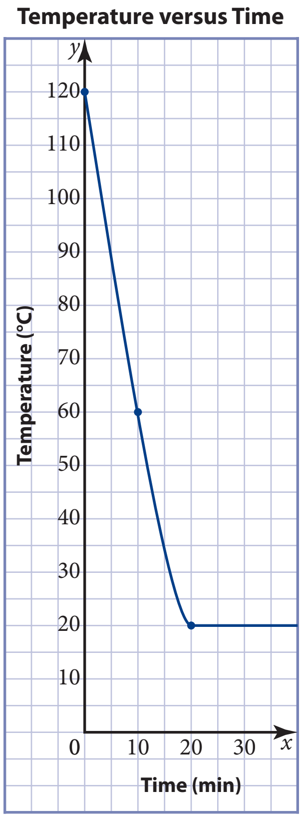
a) Copy and complete the table to show the number of bacteria at the end of every 12 h for one week.

Time (h)	Number of Bacteria (1000s)
12	1000
24	

b) Make a scatter plot of your data, with time as the independent variable.

c) When will the number of bacteria reach 1 000 000 000? Explain how you know.

8. A student in a chemistry laboratory heated a liquid chemical to 120°C and let it cool at room temperature to 60°C. This took 10 min. The graph shows the temperature of the chemical (in degrees Celsius) versus time (in minutes).



a) Describe the shape of the graph.

b) How long did it take for the chemical to cool to 30°C?

c) The temperature of the chemical will stabilize when it reaches room temperature. What is the temperature in the laboratory?

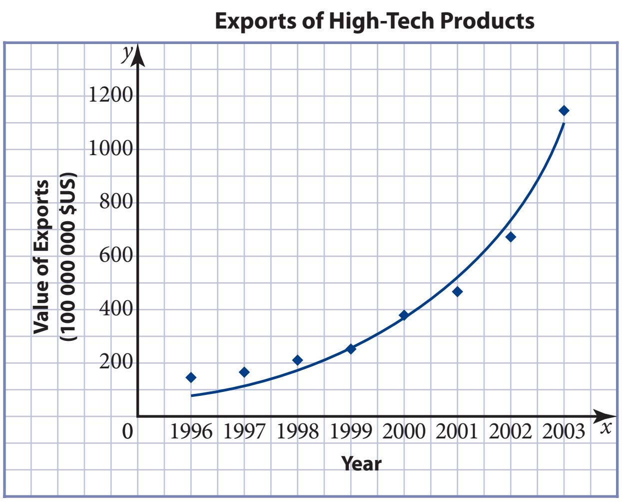
9. A photocopier is set to reduce an image to 90% of its original size.

a) If you make a copy of the reduced image and reduce it to 90%, what percent of the original is the second image?

b) How many times would you have to reduce the image to 90% for it to be reduced by at least 50% of the original? Explain.

Literacy Connect

10. The graph shows a scatter plot and a curve of best fit of total high-tech exports from China, in hundreds of millions of US dollars, between 1996 and 2003.



a) Do you think the graph is an example of exponential growth? Explain.

b) Assume the trend continued. What was the value of high-tech exports in 2004? What will it be in 2010?

c) Do you think this trend will continue? Explain.

Achievement Check

11. Mrs. Lynn's math students wanted to know how much homework was expected in her course. She suggested they spend 2 s on homework on the first night then double the time spent on homework every night throughout the course.

a) How much time would have been spent on homework at the end of the first week of classes?

b) How much time would have been spent on homework at the end of the first month of classes? (Assume there are 20 school days in a month.)

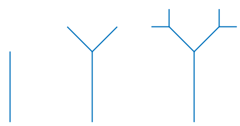
c) Estimate when the time spent on homework is 1 h.

d) Estimate when the time spent on homework is one day.

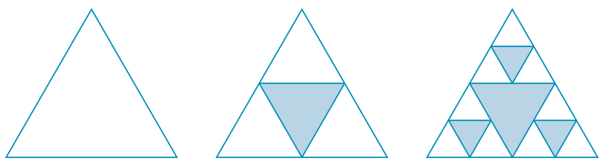
Extend C

12. Fractals are geometric figures created by repeatedly applying the same drawing process. Describe how each fractal was created. Explain how these fractals can be considered examples of exponential growth.

a)



b)



7.4 Exponential Relations

How do the graphs of linear, quadratic, and exponential relations compare?

In this section, you will learn how to identify these relations from tables of values and from their graphs.

Investigate

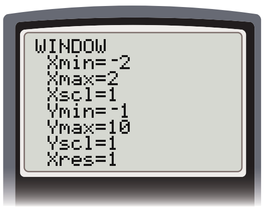
Tools

- graphing calculator

Graphs of Exponential Relations

Method 1: Use a Graphing Calculator

1. Use a graphing calculator to graph the relations $y = 2^x$ and $y = 4^x$. Use the window settings shown.



2. What points do the graphs have in common?

3. Describe the shape of each graph on the left and the right of the y-axis. How are the graphs similar? How are they different?

4. Predict the shape of each graph and the y-intercept of $y = 5^x$ and $y = 1.5^x$. Check with a graphing calculator.

5. **Reflect** Describe the graph of an exponential relation of the form $y = b^x$, for $b > 1$.

Method 2: Graph by Hand

1. Create a table of values for each relation.
   - $y = 2^x$ • $y = 4^x$

   Use $x = -2, -1, 0, 1, 2$.

2. Plot the data for each relation on the same set of axes. Use a different colour for each data set and join each set of points with a smooth curve.

3. Which points do the graphs have in common?

4. Describe the shape of each graph to the left and the right of the y-axis. How are the graphs similar? How are they different?

5. Predict the shape of each graph and the y-intercept of $y = 5^x$ and $y = 1.5^x$. Check by graphing.

6. **Reflect** Describe the graph of an exponential relation of the form $y = b^x$, for $b > 1$.

Example 1

Exponential Relations

a) Make a table of values and graph each relation.

i) $y = 2^x$

ii) $y = 3^x$

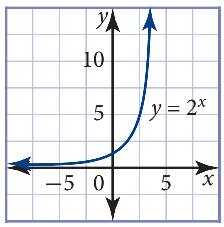
iii) $y = \left(\frac{1}{2}\right)^x$

b) Describe the similarities and differences between the graphs.

Solution

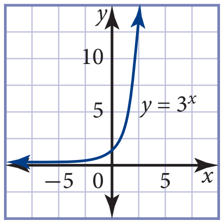
a) i)

x	y
−3	$\frac{1}{8}$
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16



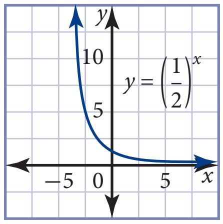
ii)

x	y
−3	$\frac{1}{27}$
−2	$\frac{1}{9}$
−1	$\frac{1}{3}$
0	1
1	3
2	9
3	27
4	81



iii)

x	y
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$



b) The graphs of $y = 2^x$ and $y = 3^x$ are almost horizontal on the left and become steeper toward the right. The graph of $y = \left(\frac{1}{2}\right)^x$ is almost horizontal on the right and becomes steeper toward the left.
All graphs have a y-intercept of 1. None have an x-intercept.
The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ are reflections of each other in the y-axis.

Example 2

Population Growth

Ontario's population is projected to grow exponentially based on the relation $P = 11\ 000\ 000(1.0112)^n$, where P is the estimated population and n is the number of years after 1996. The formula is expected to be valid until 2031.

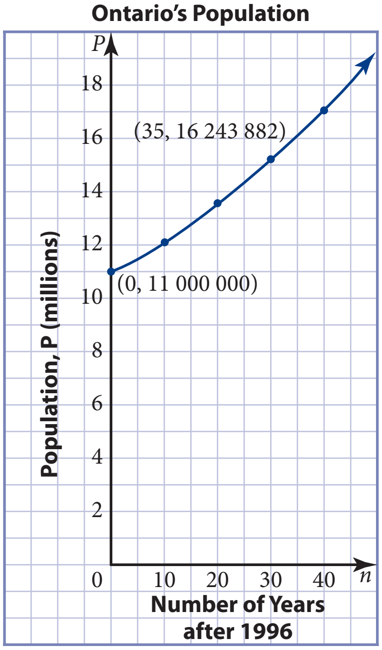
a) Sketch a graph of this relation.

b) What was Ontario's population in 1996? Show this on the graph.

c) What is the projected population for Ontario in 2031?

Solution

a)



Number of Years After 1996	Population
0	11 000 000
10	12 295 985
20	13 744 657
30	15 364 008
40	17 174 145

b) In the relation, n is the number of years after 1996. So for 1996, $n = 0$.

$$\begin{aligned} P &= 11\ 000\ 000(1.0112)^{0} \\ &= 11\ 000\ 000 \times 1 \\ &= 11\ 000\ 000 \end{aligned}$$

In 1996, the population of Ontario was about 11 000 000 people.

c) The year 2031 is 35 years after 1996, so $n = 35$.

$$\begin{aligned} P &= 11\ 000\ 000(1.0112)^{35} \\ &= 11\ 000\ 000 \times 1.476\ 716\ 518 \\ &= 16\ 243\ 881.57 \end{aligned}$$

The projected population for Ontario in 2031 is 16 243 882 people.

Example 3

Compare Linear, Quadratic, and Exponential Relations

a) Make a table of values for each relation. Use $x = -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$. Use first and second differences to determine whether each relation is linear, quadratic, or exponential. Find the ratio of successive y-values for each relation. Compare the results.

i) $y = 2x$ **ii)** $y = x^2$ **iii)** $y = 2^x$

b) Graph each relation on the same set of axes.

c) Describe each graph. Identify the y-intercept and any maximum or minimum point.

Solution

a) i) $y = 2x$

x	y	First Differences	Ratio of Successive y-values
−4	−8		
		2	$\frac{3}{4}$
−3	−6		
		2	$\frac{2}{3}$
−2	−4		
		2	$\frac{1}{2}$
−1	−2		
		2	0
0	0		
		2	undefined
1	2		
		2	2
2	4		
		2	$\frac{3}{2}$
3	6		
		2	$\frac{4}{3}$
4	8		

← $\frac{-6}{-8} = \frac{3}{4}$

- The first differences are constant so this is a linear relation. The ratios of successive y-values are not equal.

ii) $y = x^2$

x	y	First Differences	Second Differences	Ratio of Successive y-values
−4	16			
		−7		$\frac{9}{16}$
−3	9		2	
		−5		$\frac{4}{9}$
−2	4		2	
		−3		$\frac{1}{4}$
−1	1		2	
		−1		0
0	0		2	
		1		undefined
1	1		2	
		3		4
2	4		2	
		5		$\frac{9}{4}$
3	9		2	
		7		$\frac{16}{9}$
4	16			

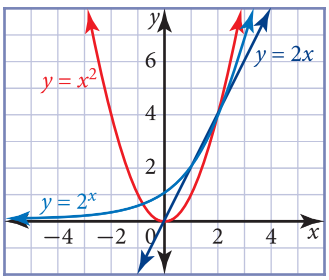
- The first differences are not constant but the second differences are, so this is a quadratic relation. The ratios of successive y-values are not equal.

iii) $y = 2^x$

x	y	First Differences	Second Differences	Ratio of Successive y-values
-4	$\frac{1}{16}$			
-3	$\frac{1}{8}$	$\frac{1}{16}$		2
-2	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	2
-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	2
0	1	$\frac{1}{2}$	$\frac{1}{4}$	2
1	2	1	$\frac{1}{2}$	2
2	4	2	1	2
3	8	4	2	2
4	16	8	4	2

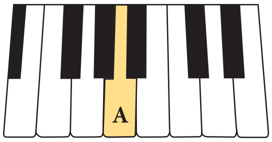
- The first and second differences are not constant, but the ratios of successive y-values are equal. This is an exponential relation.

b)



c)
- The graph of $y = 2x$ is a line with a slope of 2 and y-intercept 0. There is no maximum or minimum point.
- The graph of $y = x^2$ is a parabola with y-intercept 0. The minimum point is (0, 0).
- The graph of $y = 2^x$ is a curve that approaches a horizontal line on the left and becomes increasingly steep towards the right. The y-intercept is 1. There is no maximum or minimum point.

Example 4



Musical Scale

Middle A on a piano is known as A4. Its sound wave has a frequency of 440 cycles per second, also written as 440 Hertz (Hz). The table shows the frequencies of each of the eight A-notes on a piano.

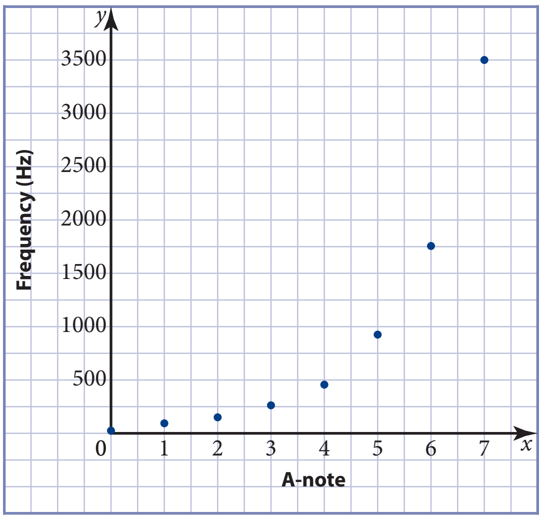
A-note	0	1	2	3	4	5	6	7
Frequency (Hz)	27.5	55	110	220	440	880	1760	3520

Show that the relationship between the A-notes on a piano and their frequencies can be modelled using exponential growth.

Solution

First, graph the relation.

Piano A-note Frequency



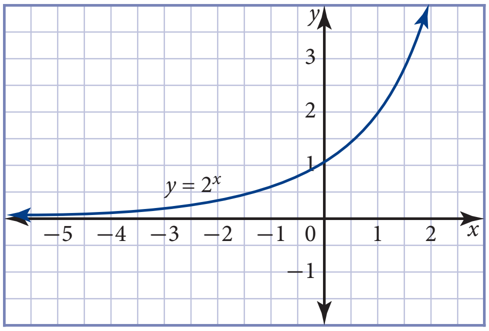
The graph of the relation curves upward and increases more and more rapidly from left to right. The curve is constantly increasing. This curve might be part of a parabola. Use first and second differences to check.

A-note	0	1	2	3	4	5	6	7
Frequency (Hz)	27.5	55	110	220	440	880	1760	3520
First Differences		27.5	55	110	220	440	880	1760
Second Differences			27.5	55	110	220	440	880
Ratio of Successive *y*-values		2	2	2	2	2	2	2

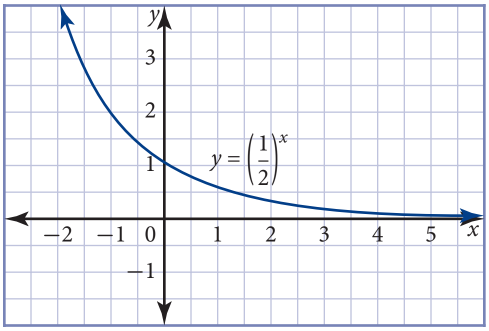
Neither the first nor the second differences are constant, so the relation is neither linear nor quadratic.
The ratios of successive y-values are all equal. Therefore the relationship between the A-notes on a piano and their frequencies is exponential.

Key Concepts

- A relation of the form $y = b^x$, where $b > 0$ and $b \neq 1$, is exponential.
- If $b > 1$, moving left to right, the graph increases very slowly for negative x-values and increases more rapidly for positive x-values. The graph is almost horizontal on the left and very steep on the right.



- If $0 < b < 1$, moving left to right, the graph decreases very rapidly for negative x-values and decreases more slowly for positive x-values. The graph is almost horizontal on the right and very steep on the left.



- The y-intercept is 1 and there is no x-intercept.
- The growth or decay factor is the base of the power, b.

Discuss the Concepts

D1. The graph of an exponential relation does not have an x-intercept. Explain why, using the graph from Example 3.

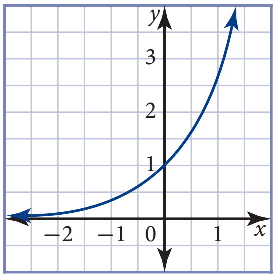
D2. Compare the graphs of $y = x^2$ and $y = 2^x$ in Example 3. Describe what happens to the value of y for each graph when x changes from a positive number to a negative number.

Practise A

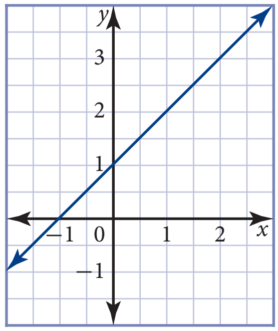
For help with question 1, refer to Example 3.

1. Identify the type of growth (linear, quadratic, exponential) illustrated by each graph. Justify your answers.

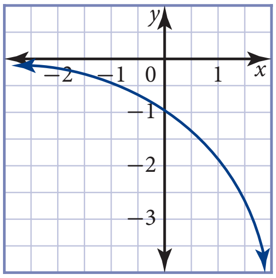
a)



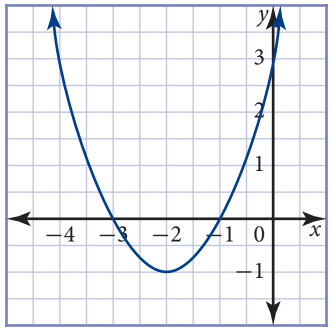
b)



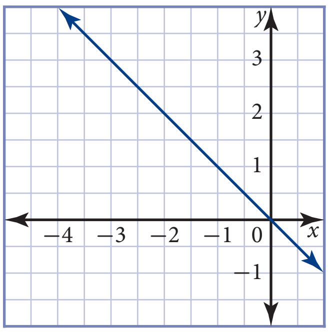
c)



d)



e)



For help with questions 2 to 5, refer to Example 1.

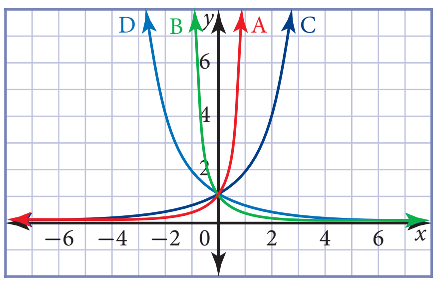
2. Identify which graph represents each relation. Justify your response.

a) $y = 2^x$

b) $y = 10^x$

c) $y = \left(\frac{1}{2}\right)^x$

d) $y = (0.1)^x$



3. a) Sketch each graph. Use a graphing calculator to check.
   i) $y = 2^x$ ii) $y = 2(2^x)$
   iii) $y = 3(2^x)$ iv) $y = 4(2^x)$
   b) Describe the role of a in $y = a(b^x)$.

4. Make a table of values for each relation. Sketch each pair of relations on the same set of axes. Use a graphing calculator to check.
   a) $y = 3^x$ $y = 2(3^x)$
   b) $y = \left(\frac{1}{2}\right)^x$ $y = 2\left(\frac{1}{2}\right)^x$
   c) $y = (0.4)^x$ $y = 0.3(0.4)^x$

5. Make a table of values for each relation. Graph each pair of relations on the same set of axes. Use a graphing calculator to check.
   a) $y = 2^x$ $y = 2^{3x}$
   b) $y = 10^x$ $y = 10^{\frac{x}{2}}$
   c) $y = \left(\frac{1}{2}\right)^x$ $y = \left(\frac{1}{2}\right)^{\frac{x}{4}}$
   d) $y = 2^x$ $y = (3)2^{\frac{x}{5}}$

For help with question 6, refer to Example 2.

6. York Region's population, P, is projected to grow until 2031 based on the relation $P = 610\ 000(1.029)^n$, where n is the number of years after 1996.
   a) Sketch a graph of this relation.
   b) What is the P-intercept? What does it represent?
   c) What is the projected population of York Region in
      i) 2015?
      ii) 2031?

7. A pressure reader is used to measure the sound intensity of a bell. The relation $P = 200(0.5)^t$ estimates the sound pressure, P, in pascals, after t seconds.
   a) Sketch a graph of this relation.
   b) What is the P-intercept? What does it represent?
   c) What was the sound pressure after
      i) 1 s?
      ii) 2 s?

Apply B

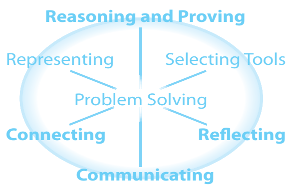
8. Between 1996 and 2006, the population of Toronto grew from 2 459 700 to 2 607 600. The population of Peel Region grew from 878 800 to 1 215 300. The populations, P, can be estimated using the relations:

$P_{\text{Toronto}} = 2\ 459\ 700(1.0058)^n$ $\qquad$ $P_{\text{Peel}} = 878\ 800(1.033)^n$

where n is the number of years after 1996.

a) Make a table of values of each population for 10 years after 1996 and sketch a graph for each relation.

b) Compare the growth rates. How do the growth rates affect the graphs?



9. Which model (linear, quadratic, or exponential) would best describe each situation? Why?

a) a car slowing down by $\frac{1}{4}$ of its speed for every second that elapses

b) the height of a stone falling from the top of a cliff

c) a motorcyclist speeding up by 4 km/h each second

d) the number of bacteria doubling every 3 h

e) the path of a basketball when tossed into the air

f) the maximum height of each bounce of a bouncing ball

Chapter Problem

10. The sound pressure, in micropascals, is the air pressure exerted by sound waves on objects such as your ear drums. To convert decibels (dB) to sound pressure (P), you can use the relation $P = 20 \times 10^{\frac{dB}{20}}$.

a) Plot a graph of this relation. Use dB-values from 0 to 160, in steps of 20.

b) Normal conversation measures 60 dB. Sound at a rock concert can reach 120 dB. Compare the sound pressures for these two situations.

c) If the sound level reaches 160 dB, it can perforate your eardrums. What sound pressure will cause this?

11. The sound wave for each note on a piano has a different frequency. A full octave on a piano from note C4 to C5 is shown.

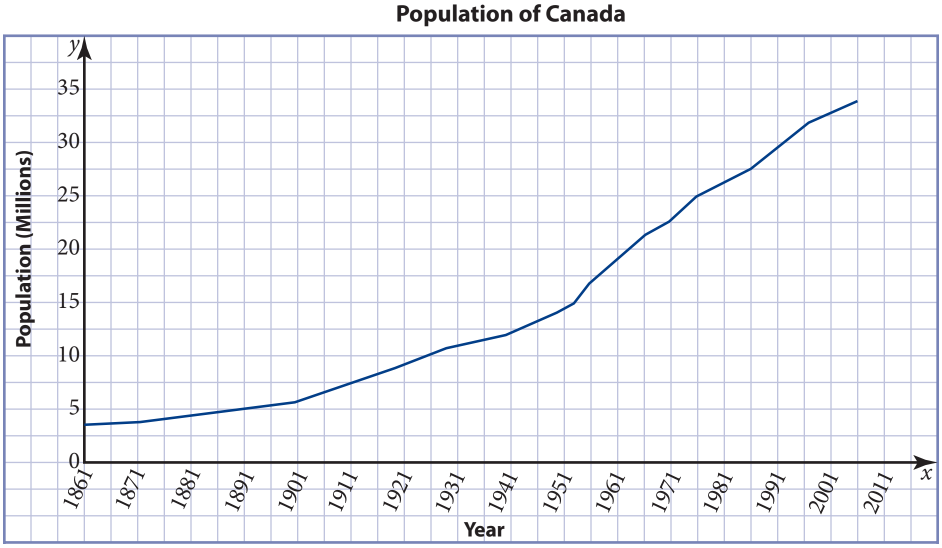
Note	C4	C#	D	D#	E	F	F#
Frequency (Hz)	261.6	277.2	293.7	311.1	329.6	349.2	370.0

Note	G	G	A	A	B	C5
Frequency (Hz)	392.0	415.3	440.0	466.2	493.9	523.2

a) Show that the relationship between the notes in an octave and their frequency can be modelled using exponential growth. How does this compare to the results in Example 4?

b) Graph the relation using a graphing calculator. (Hint: Let C4 represent $x = 1$, C# represent $x = 2$, and so on.) Compare your graph to the graph of a linear relation.

12. The graph shows the population of Canada from 1861 to 2001.



a) What was the approximate annual rate of change in population from 1861 to 1871?

b) What was the approximate annual rate of change in population from 1951 to 1961?

c) What was the approximate annual rate of change in population from 1991 to 2001?

d) What type of relation is represented by this graph? Explain.

e) What was the approximate **doubling time** of Canada's population?

f) Predict when Canada's population will grow to about 64 million people.

doubling time

- time required for a quantity to double in size, number, or mass

Literacy Connect

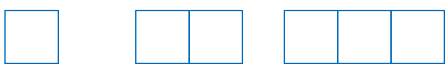
13. Consider each pattern of diagrams.

i) Describe the pattern.

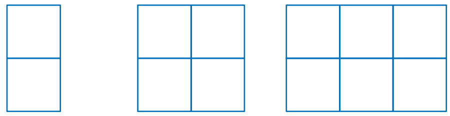
ii) Continue each pattern for two more diagrams.

iii) What type of relation is represented?

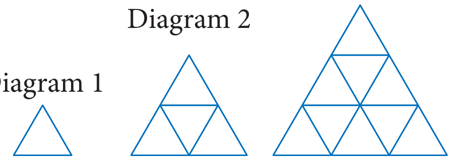
a) Diagram 1 Diagram 2 Diagram 3



b) Diagram 1 Diagram 2 Diagram 3



c) Diagram 1 Diagram 2 Diagram 3



Extend C

14. Air temperature and atmospheric pressure are important factors in predicting the weather. Atmospheric pressure, measured in pascals (Pa), is the weight of the gas molecules that make up the atmosphere. The table shows the temperature and atmospheric pressure at various altitudes above a specific location on Earth.

Altitude (m)	Temperature (°C)	Atmospheric Pressure (Pa)
0	15.00	10.13
1000	5.16	8.99
2000	−4.68	7.95
3000	−14.52	7.01
4000	−24.36	6.16
5000	−34.20	5.40
6000	−44.04	4.71
7000	−53.88	4.11
8000	−63.72	3.55

a) Compare the changes in temperature and atmospheric pressure as the altitude increases.

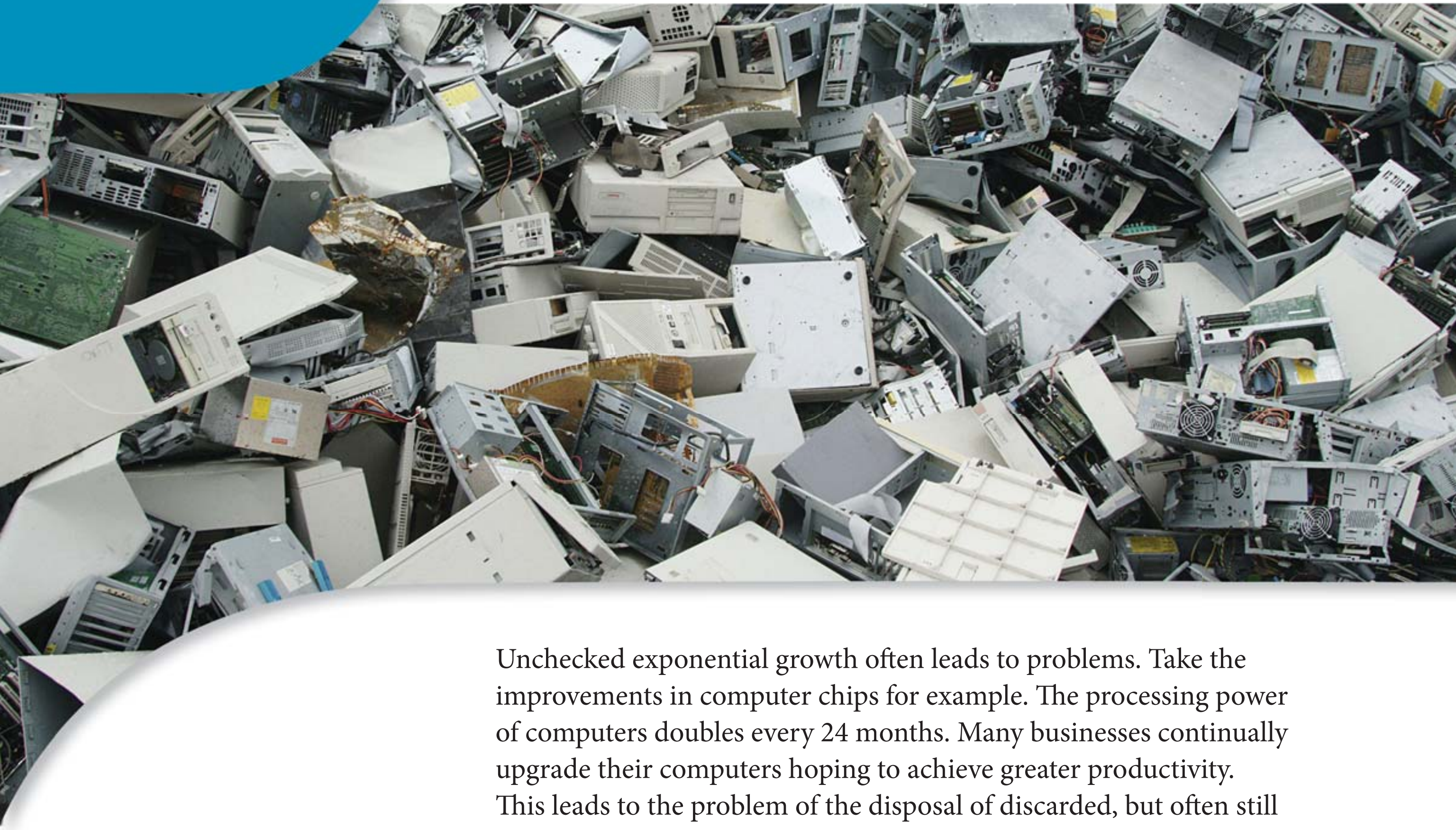
b) Compare the types of growth for temperature and atmospheric pressure as the altitude increases.

c) Why does the atmospheric pressure and temperature decrease as the altitude increases?

15. **a)** Explain why $y = 52(0.5)^x$ can be rewritten as $y = 52(0.5)(0.5)^{x-1}$.

b) Rewrite $y = 52(0.5)(0.5)^{x-1}$ in the form $y = a \times b^{x-1}$.

7.5 Modelling Exponential Growth and Decay



Unchecked exponential growth often leads to problems. Take the improvements in computer chips for example. The processing power of computers doubles every 24 months. Many businesses continually upgrade their computers hoping to achieve greater productivity. This leads to the problem of the disposal of discarded, but often still serviceable, computers.

Investigate 1

Tools

• graphing calculator

Bacterial Growth

You will use linear, quadratic, and exponential regression to find the equation that best represents a given set of data. Each time, you will record the equation and the value of r^2, the coefficient of determination. The value of r^2 measures how well the equation fits the data. The closer r^2 is to 1, the better the equation fits the data.

A biology student is studying the population growth of a bacterial culture. The mass of the culture is measured every hour, but the initial population at time zero is unknown. The table shows the observations for the first 7 h.

Time (h)	0	1	2	3	4	5	6	7
Mass (mg)	■	10	21	43	82	168	320	475

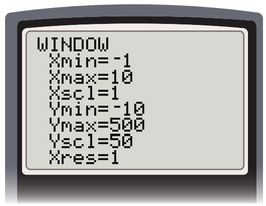
Before starting the activity, clear all lists and equations.

1. a) Enter the data into lists.

- Press [STAT]. Select **1:Edit**. Enter the time data (from 1 h to 7 h) into list **L1** and the corresponding population data into list **L2**.

b) Make a scatter plot.

- Press [2nd] [STATPLOT]. Select **Plot1**. Ensure that it is turned **On**. Set the graph to scatter plot, the **Xlist** to **L1** and the **Ylist** to **L2**.



- Press [WINDOW]. Use the window settings shown.
- Press [GRAPH]. The scatter plot will appear.

Describe the shape of the graph.

c) Perform a linear regression analysis.

- Press [2nd] [CATALOG] and scroll down to **Diagnostic On**.
- Press [ENTER] twice.
- Press [STAT]. Cursor over to the **CALC** menu then select **4:LinReg(ax+b)**.
- Press [2nd] [L1] [,] [L2] [,].
- Press [VARS]. Cursor over to the **Y-VARS** menu, then select **1:Function**.
- Press [ENTER] twice. The equation of the line of best fit will be displayed. Record the equation, and the coefficient of determination, r^2, in a table like the one shown.

Method	Equation	Coefficient of Determination, r^2	Mass at Hour 8	Mass at Hour 9
Linear regression				

d) Press [Y=] to see the equation of the line of best fit in the equation editor.

- Press [GRAPH] to see the graph of the line of best fit. Explain why this is an unsatisfactory model for this problem.

2. Perform a quadratic regression analysis.

- Press [Y=] [CLEAR].
- Press [STAT], then from the **CALC** menu, select **5:QuadReg**.
- Press [VARS]. Cursor over to the **Y-VARS** menu, then select **1:Function**.
- Press [ENTER] twice. Add another row to your table for quadratic regression. Record the equation, and the coefficient of determination, r^2.
- Press [GRAPH] to see the graph of the quadratic equation.

3. a) Perform an exponential regression analysis. Record the equation and the coefficient of determination.

- Press [STAT], then from the **CALC** menu, select **0:ExpReg**.
- Press [VARS]. Cursor over to the **Y-VARS** menu, then select **1:Function**.
- Press [ENTER] twice.
- Press [GRAPH] to see the graph of the exponential equation.

b) Use this model to estimate the initial mass.

c) **Reflect** Is the exponential model a better model than those you used in questions 1 and 2? Explain.

4. Which equation fits the data best? worst? Explain how you know.

5. Examine the data for hours 8 and 9 for all three methods. What happens to the accuracy of the linear equation as time progresses?

Investigate 2

Tools

- graphing calculator
- CBR™
- basketball or volleyball

Bouncing Ball

You will work with a partner to determine a model that fits the data for a bouncing ball.

a) Connect a calculator-based ranger (CBR™) to a graphing calculator. Clear all lists.

b) Go to the Ball Bounce application.

- Press [APPS], select **CBL/CBR**, and press [ENTER]. Then, select **3:Ranger**.
- Press [ENTER]. Select **3:APPLICATIONS**. Select **1:METERS**, then **3:BALL BOUNCE**.

c) Follow the instructions on the screen. Hold the CBR™ at a height of at least 1.5 m, with the sensor pointing down. Drop the ball from a point about 0.5 m directly below the CBR™. Press the trigger on the CBR™ and allow the ball to bounce at least five times.

d) When the measurements are complete, a graph will appear on the calculator screen. What does the horizontal axis of this graph represent? What does the vertical axis represent?

e) Describe the relationship between the bounce number and height of the bounce. Describe the shapes of the curves. What is the cause of the shape of each curve?

f) Find the maximum height the ball reaches on each bounce.

- Press [TRACE] and move the cursor to the top of the curve representing each bounce. On a piece of paper, record the coordinates of these points in a table with the headings "Time (s)" and "Bounce Height (m)."

g) Divide each maximum bounce height by the previous one. What is the friction of the floor causing the ball to do?

h) Enter data into lists 3 and 4.

- Press STAT and select **1:Edit**, then press ENTER. Enter the times into **L3** and the bounce heights into **L4**.

i) Plot these coordinates using **STAT PLOT**.

- Press 2nd [STATPLOT], then select **2:Plot2**.
- Press ENTER, and make sure Plot 2 is **On**. Select the line graph icon. Enter **L3** for **XList** and **L4** for **YList**.
- Then, press GRAPH.

j) Describe the shape of the graph. If you were to extend the graph to the right, what would happen?

k) **Reflect** Is the relationship between time and bounce height exponential? Explain.

Exponential Growth and Decay

Many situations can be modelled with an exponential relation that represents exponential growth or exponential decay. The formula or relation $P = I(b)^t$ can be used to determine the population size, P, after time t, where I is the initial population and b is the growth factor (if $b > 1$) or the decay factor (if $0 < b < 1$).

Example 1

Animal Population

In a national park, a wolf population increased by a growth factor of 1.078 per year over a ten-year period, beginning in 1997. The formula $P = 124(1.078)^n$ modelled the wolf population P after n years.

a) Use technology to graph the relation.

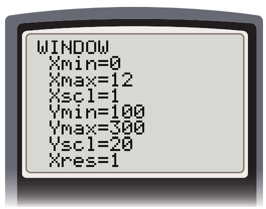
b) What was the wolf population in 1997?

c) What was the wolf population in 2007?

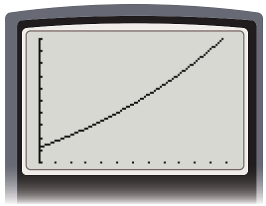
Solution

a) Press 2nd [STATPLOT]. Select **4:PlotsOff**.

- Press ENTER.
- Press Y=. If necessary, clear all equations.
- Type 124 × 1.078 ^, and then press X,T,θ,n.
- Press WINDOW. Use the window settings shown.

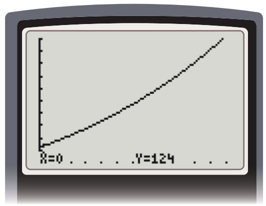


- Press GRAPH.



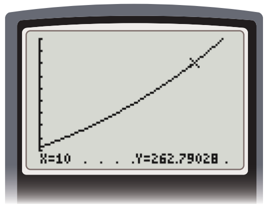
b) Press 2nd [CALC]. Select **1:value**.
- Press ENTER, then enter 0 for **X**=.
  Press ENTER.
  (Since the values on x-axis begin in 1997, $x = 0$ represents 1997.)

In 1997, the wolf population was 124 wolves.



c) Repeat part b) but enter 10 for **X**=. (Since the values on the x-axis begin in 1997, $x = 10$ represents 2007.)

In 2007, the wolf population was 263 wolves.



Example 2

Light Intensity

A sheet of translucent glass 1 mm thick reduces the intensity of the light passing through it. Light intensity is further reduced as more sheets of glass are placed together, as shown in the table.

Number of Glass Sheets	0	1	2	3	4	5	6	7	8
Light Intensity (%)	100	89.1	79.4	70.7	63.0	56.1	50.0	44.5	39.7

a) The reduction rate of a sheet of glass is the percent by which the light intensity is reduced by adding a sheet of glass to a viewing panel. What is the light intensity reduction rate of a single sheet of glass? Express your answer as a percent.

b) How many sheets of glass are needed to reduce the light intensity by one half?

c) Use a graphing calculator to graph this relation.

d) Use your graph to determine how many sheets of glass are needed to reduce the light intensity to about 25%.

Solution

a) To find the reduction rate, subtract two successive light intensities and divide by the greater of the two.

$(100 - 89.1) \div 100 \times 100\% = 10.9\%$

$(89.1 - 79.4) \div 89.1 \times 100\% = 10.9\%$

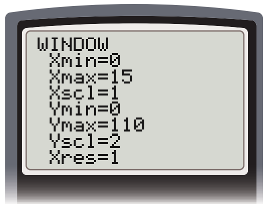
and so on.

The reduction rate of a single sheet of glass is 10.9% of the previous light intensity.

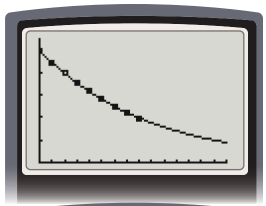
b) From the table, six sheets of glass would reduce the light intensity by 50%.

c) Press 2nd [MEM] to clear all lists. Then press ENTER.

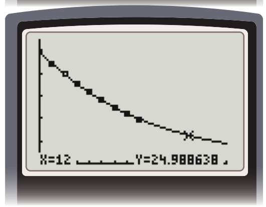
- Press STAT then select **1:Edit**. Enter the number of glass sheets into list **L1**, and light intensity into list **L2**.
- Press 2nd [STATPLOT]. Select **Plot1**. Ensure that it is turned **On**. Set the graph to scatter plot, the **XList** to **L1**, and the **YList** to **L2**. Check that **Plot2** and **Plot 3** are turned **Off**.
- Press WINDOW and adjust the variables as shown.



- Press STAT, then from the **CALC** menu, select **0:ExpReg**.
- Press VARS. Cursor over to the **Y-VARS** menu, then select **1:Function**.
- Press ENTER twice.
- Press GRAPH.



d) Press TRACE. Use the cursor keys to move the point shown on the graph until the value of **Y** is as close as possible to 25. The calculator returns the value 12.



To reduce the intensity to 25%, 12 sheets of glass are needed.

Key Concepts

- Experimental data or data from secondary sources can be graphed as a scatter plot and modelled using technology.
- Exponential regression can be used to generate the equation for an exponential relation.
- The equation $P = I(b)^t$ can be used to model an exponential relation, where P is the current population, I is the initial population, b is the growth factor (if $b > 1$) or decay factor (if $0 < b < 1$), and t is the time.
- The coefficient of determination, r^2, indicates how well an equation fits the data. If r^2 is close to 1, the equation fits the data well.

Discuss the Concepts

D1. Explain why population growth, such as a fox population that grows by 2% per year, represents exponential growth.

D2. Pollution in a lake has reduced the clarity of the water by reducing light intensity by 20%/m of depth. The formula $I = 100(0.8)^d$ can be used to calculate the light intensity, I, at a depth of d metres.

a) Explain how the formula relates to a 20% reduction in water clarity.

b) Predict the shape of the graph of this relation. Justify your prediction.

Practise A

For help with questions 1 and 2, refer to Example 1.

1. Cells in a culture are growing by a factor of 3.45 per day. The number of cells in the culture, N, can be estimated using the formula $N = 1000(3.45)^d$, where d is the number of days.

a) Use technology to plot a graph of this relation.

b) How many cells does this culture begin with?

c) How many cells would there be after 1 day?

d) How many cells would there be after 5 days?

2. A deer population is declining by 2.2% per year. The population can be modelled using the formula $P = 240(0.978)^n$, where P is the population after n years.

a) Use technology to plot a graph of this relation.

b) What is the current deer population?

c) What will be the expected deer population after 8 years?

For help with question 3, refer to Example 2.

3. Caffeine is present in coffee, tea, chocolate, and other foods and beverages. This chemical is eliminated from the human body over time. The table shows the mass of caffeine remaining in an average-sized person after drinking a cup of coffee containing 130 mg of caffeine.

Time (h)	0	1	2	3	4	5	6	7
Mass of Caffeine (mg)	130	113.1	98.4	85.6	74.5	64.8	56.4	49.0

a) What percent of the caffeine is eliminated from a person's body per hour?

b) Use a graphing calculator to graph this relation.

c) Use the graph to estimate how long it would take for the mass of caffeine to be reduced to less than 10 mg.

d) Use the graph to estimate how long it would take for the mass of caffeine to be reduced to 0 mg.

Apply B

4. A can of cola contains 45.6 mg of caffeine and a typical chocolate bar contains 31 mg of caffeine.

a) Use the rate of change from question 3 a) to make a table of values showing the mass of caffeine remaining in an average-sized person after consuming

i) a can of cola **ii)** a chocolate bar

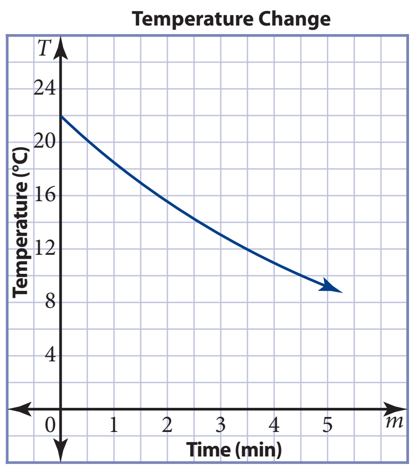
b) Use a graphing calculator to graph each relation.

c) Use the graph to estimate how long it would take for the mass of caffeine to be reduced to less than 10 mg for each product.

d) Pose and solve a problem relating to this situation.

5. A large number of ice cubes were added to a pitcher of water at room temperature, 22°C. The graph shows the temperature of the water, *T*, in degrees Celsius, over a period of time, *m*, in minutes.

a) How long did it take for the temperature to fall to 18°C?



b) How long did it take for the temperature to fall to 16°C?

c) Is it valid to use this graph to extrapolate past 5 min? Explain.

d) Pose and solve a problem relating to this situation.

Literacy Connect

The amplitude of a spring is half the distance between rest and the maximum height it reaches with each oscillation (up and down).

6. A motion detector recorded the amplitude of an oscillating spring. The table shows the results over a 5-min period.

Time (s)	0	30	60	90	120	150	180	210	240	270	300
Amplitude (cm)	9.2	7.2	5.5	4.4	3.4	2.6	2.0	1.6	1.3	1.0	0.7

a) Do the results of this experiment demonstrate exponential decay? Explain.

b) What is the approximate rate of growth or decay?

c) How long would it take for the oscillations to become indiscernible (less than 0.5 cm)?

7. The Mauna Loa Observatory in Hawaii records the level of carbon dioxide (CO_2) in the atmosphere to help study the effects of burning fossil fuels. The April readings, in parts per million (ppm), from 1959 to 2002 are shown.

Year	CO_2 (ppm)	Year	CO_2 (ppm)	Year	CO_2 (ppm)	Year	CO_2 (ppm)
1959	317.71	1970	328.13	1981	342.51	1992	359.15
1960	319.03	1971	327.78	1982	343.56	1993	359.46
1961	319.48	1972	329.72	1983	344.94	1994	361.26
1962	320.58	1973	331.50	1984	347.08	1995	363.45
1963	321.39	1974	332.65	1985	348.35	1996	364.72
1964	322.23	1975	333.31	1986	349.55	1997	366.35
1965	322.13	1976	334.58	1987	350.99	1998	368.61
1966	323.70	1977	336.07	1988	353.59	1999	371.14
1967	324.42	1978	337.76	1989	355.42	2000	371.66
1968	325.02	1979	338.89	1990	356.20	2001	372.87
1969	326.66	1980	340.77	1991	358.60	2002	374.86

a) Describe what is happening to the level of CO_2 in the atmosphere over time.

b) Compare the total and percent increases in CO_2 from 1961 to 1962 with 1981 to 1982 and with 2001 to 2002.

c) Compare the total and percent increases in CO_2 in each of the decades: 1960s, 1970s, 1980s, and 1990s.

d) Which of the answers to parts b) and c) gives a better indication of the exponential growth of CO_2 levels? Explain.

e) Pose a problem relating to this table. Ask a classmate to solve the problem.

8. Many animals and fish show an exponential relationship between length and mass. The table shows the lengths and masses of rainbow trout taken from different rivers in the Muskoka area.

Length (mm)	Mass (g)	Length (mm)	Mass (g)
390	660	305	303
368	581	335	410
385	609	317	335
360	557	351	506
346	433	368	605
438	840	326	353
392	623	270	209
324	387	359	476
360	479	347	432
413	754	259	202
276	235	247	184
334	406	280	248
332	383	265	223
324	353	318	340
337	363	305	303
343	390	335	410
318	340	317	335

a) Use graphing technology to create a scatter plot and an exponential curve of best fit.

b) Is the relationship between fish length and mass exponential? Justify your answer.

c) Estimate the mass of a trout that has a length of 450 mm.

d) How might people use this relationship?

9. Research data on an animal of your choice.

a) Find data on two measures for a sample of animals. Display the data in a table.

b) Use graphing technology to graph the data to create a scatter plot and an exponential curve of best fit.

c) Is the relationship between the two measures exponential? Justify your answer.

d) How might researchers use this relationship?

Achievement Check

10. Weather satellites and space probes can be powered by thermoelectric generators. The source of power for these generators is the energy produced by radioactive material that decays. The energy is in the radioisotopes that make up the material. The power output of the radioisotopes is *P*, in watts, and *t* is the half-life of the material, in years. Different radioactive materials can be used.

Time, *t* (years)	Power, *P* (watts)	Time, *t* (years)	Power, *P* (watts)
0	50.0	150	27.4
25	45.2	175	24.8
50	40.9	200	22.4
75	37.0	225	20.3
100	33.5	250	18.4
125	30.3	275	16.6

a) Use graphing technology to create a scatter plot of the data.

b) Find an equation that models the data using an exponential regression.

c) If the equipment in the satellite needs at least 15 W of power to function, for how long can the satellite operate before needing to be recharged?

Extend C

11. The projected populations, *P*, of Metropolis and of Gotham City can be modelled by $P_{Metropolis} = 117\,000(1.018)^n$ and $P_{Gotham} = 109\,000(1.028)^n$, where n is number of years after 2006. Use a graphing calculator to determine when the populations will be the same.

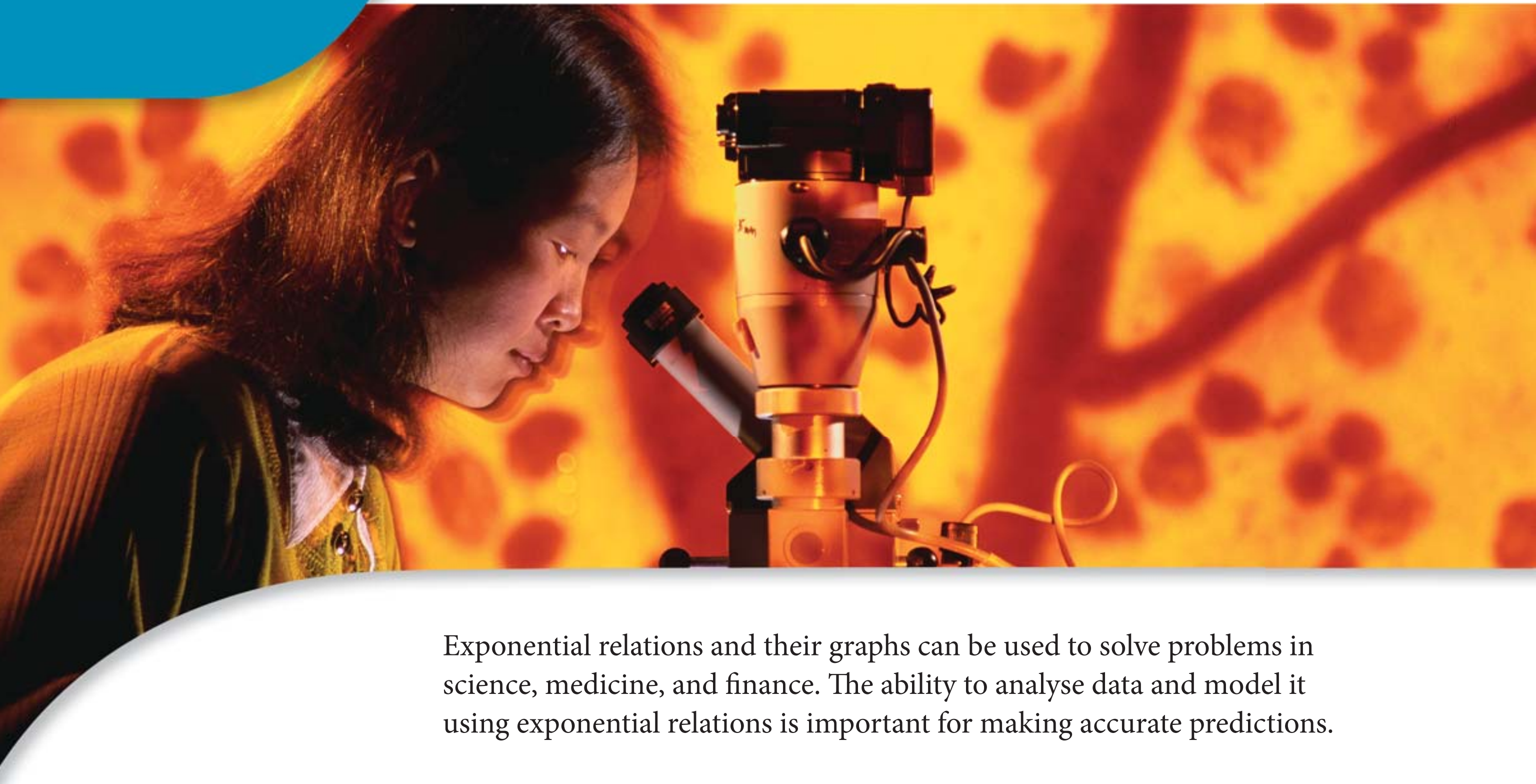
12. The world's population in 1980 was about 4.5 billion. Suppose the population increased at a rate of 2% per year since then.

a) Write an exponential relation that models the problem. Explain what each variable represents and how you determined the rate value.

b) What will be the world's population in 2015?

c) What was the population in 1970? What assumptions have you made?

7.6 Solve Problems Involving Exponential Growth and Decay



Exponential relations and their graphs can be used to solve problems in science, medicine, and finance. The ability to analyse data and model it using exponential relations is important for making accurate predictions.

Investigate

Tools

- one die or number cube, or spinner with six equal sections (per student)

Optional:

- random number generator with six outcomes

half-life

- the time it takes for a quantity to decay or be reduced to half its initial amount

Half-Life

Radioactive decay is a process by which a substance made of unstable atomic nuclei changes, or decays, into a different substance. In this process, the core of an original atom or parent nucleus is split into two smaller or daughter nuclei. The time it takes for half of the parent nuclei to decay to daughter nuclei is called the **half-life**.

In this Investigate, your class will simulate radioactive decay. A standing student will represent a parent nucleus. A sitting student will represent a daughter nucleus. Decay is determined by the roll of a die. Each student begins standing. If they roll a 6, the student "decays" to the sitting position. Instead of measuring the time it takes for half of the substance to decay, you will count the number of rolls it takes for half the class to sit down.

Before you begin the activity, predict the number of rolls it will take for half the class to be seated. Record your prediction.

1. On the teacher's prompt, roll your die. If a 6 is rolled, "decay" to the sitting position. Record the number of students who remain standing in a table like the one shown.

Trial	Number of Students in Class	Number of Students Standing After Roll Number										
		1	2	3	4	5	6	7	8	9	10	11
1												
2												
3												
4												

2. Repeat step 1 until only half the students are standing. Record the number of rolls it took for half the students to "decay" to the sitting position in the First Half-Life column in a table like the one shown.

Trial	First Half-Life	Second Half-Life
1		
2		
3		
4		
Average		

3. Remain sitting or standing. Repeat step 1 until half the remaining students are standing, that is, until one quarter of the class are standing. Record the number of students that remain standing after each roll. Record the number of rolls it took for half the students to "decay" to the sitting position in the Second Half-Life column.

4. Repeat steps 1—3 three more times, and record the results.

5. **a)** Calculate the average for both the first and second half-lives.

b) The first and second half-lives should be the same. Explain why. They may not be the same for one or more trials. Give an explanation for this.

6. **Reflect** How accurate was the prediction you made before the investigation? What was your thinking?

7. **Reflect** Would the results of this investigation have turned out differently if the number 3 represented a decay? Explain.

Doubling

When the base of an exponential relation is 2, the relation is describing a doubling.

$P = P_0(2)^{\frac{t}{d}}$ is an example of a formula that can be used to determine the population, P, after time t, where d is the doubling time, and P_0 is the initial population.

Half-Life

When the base of an exponential relation is $\frac{1}{2}$, the relation is describing half-life.

$M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ is an example of a formula that can be used to determine the remaining mass, M, of a decaying substance after time t, where h is the half-life and M_0 is the initial mass.

Example 1

Exponential Growth

A bacterial culture began with 7500 bacteria. Its growth can be modelled using the formula $N = 7500 \times 2^{\frac{t}{36}}$, where N is the number of bacteria after t hours.

a) How many bacteria are present after 36 h?

b) How many bacteria are present after 72 h? How does this relate to the doubling time?

Solution

a) Substitute $t = 36$.

$$\begin{aligned} N &= 7500(2)^{\frac{36}{36}} \\ &= 7500(2)^1 \\ &= 15\ 000 \end{aligned}$$

There are 15 000 bacteria present after 36 h.

b) Substitute $t = 72$.

$$\begin{aligned} N &= 7500(2)^{\frac{72}{36}} \\ &= 7500(2)^2 \\ &= 30\ 000 \end{aligned}$$

There are 30 000 bacteria after 72 h.

There are two doubling periods in 72 h, so the bacteria have doubled twice in this time.

Example 2

Exponential Decay

All living organisms contain a known concentration of 1 part per trillion parts of carbon-14. Carbon-14 is a radioactive element. It is used to date ancient artefacts because it has a half-life of about 5730 years after the organism dies. The formula $C = \left(\frac{1}{2}\right)^{\frac{n}{5730}}$ is used to calculate the concentration, C, in parts per trillion, remaining n years after death.

a) What would be the concentration of carbon-14 in a piece of cloth (made from plant fibres) after 5730 years?

b) What would be the concentration of carbon-14 in an animal bone after 50 000 years? Round your answer to five decimal places.

Math Connect

One trillion is 10^{12} or 1 000 000 000 000.

Solution

a) Substitute $n = 5730$.

$$C = \left(\frac{1}{2}\right)^{\frac{5730}{5730}}$$
$$= \left(\frac{1}{2}\right)^{1}$$
$$= \frac{1}{2}$$
$$= 0.5$$

After 5730 years, the concentration of carbon-14 would be 0.5 parts per trillion.

b) Substitute $n = 50\ 000$.

$$C = \left(\frac{1}{2}\right)^{\frac{50\ 000}{5730}}$$
$$= 0.002\ 36$$

(1 ÷ 2) yˣ (50000 ÷ 5730) = =

(Remember that calculator key sequences may vary depending on your calculator.)

After 50 000 years, the concentration of carbon-14 would be about 0.002 36 parts per trillion.

Key Concepts

- Doubling time is the time it takes for a population to double in size. The relation for doubling is $P = P_0(2)^{\frac{t}{d}}$, where P represents the population, P_0 represents the initial population, t represents time, and d represents the doubling time. The base, 2, indicates doubling.
- Half-life is the time it takes for a quantity to decay to half its original amount.
  The relation for half-life is $M = M_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where M represents the final quantity, M_0 represents the initial quantity, t represents time, and h represents the half-life. The base, $\frac{1}{2}$, indicates half-life.

Discuss the Concepts

D1. Refer to the Investigate. If the half-life activity were performed with a fair coin, and tossing heads meant you sit, what would be the half-life of the class?

D2. The relation for doubling uses 2 as the base of the power and the formula for half-life uses $\frac{1}{2}$ as the base. What do you think the base would be if a population is tripling? Explain.

Practise A

For help with question 1, refer to Example 1.

1. *E. coli* is a very harmful type of bacteria that can be found in meat that is improperly stored or handled. The relation $N = N_0 \times 2^{\frac{t}{20}}$ estimates the number of *E. coli*, N, of an initial sample of N_0 bacteria after t min, at 37°C (body temperature), under optimal conditions.

a) What is the doubling time of *E. coli*?

b) If a sample of *E. coli* contains 5000 bacteria, how many will there be after 1 h?

c) If a sample of *E. coli* contains 1000 bacteria, how many will there be after 1 day?

2. The intensity of light from a luminating object decays exponentially with the thickness of the material covering it. Stage lights are often covered with gels to colour the light, but they also decrease light intensity. The relation $I = 1200\left(\frac{4}{5}\right)^n$ is used to determine the intensity of light, I, in watts per square centimetre, where n is the number of gels used. What is the intensity of light with

a) 0 gels? **b)** 1 gel? **c)** 3 gels? **d)** 5 gels?

3. Certain types of minor skin wounds heal at a rate modelled by the relation $W = W_0\left(\frac{1}{2}\right)^{0.36t}$, where W is the area of the wound currently, in square millimetres, W_0 is the initial wound area, and t is the time, in days, after the wound has been dressed. What will be the area of a 25 mm^2 wound after

a) 1 day?

b) 4 days?

Apply B

For help with questions 4 and 5, refer to Example 2.

4. The half-life of carbon-14 is 5730 years. The relation $C = \left(\frac{1}{2}\right)^{\frac{n}{5730}}$ is used to calculate the concentration, C, in parts per trillion, remaining n years after death. Determine the carbon-14 concentration in

a) an 11 460-year-old animal bone

b) a 5000-year-old map made from plant fibres

c) a 25 000-year-old fossil

5. A fifteenth-century map (made from plant fibres) indicated that the Vikings settled in Vinland in northern Newfoundland in 970 B.C.E. If this map was made in 1427, what would be the concentration of carbon-14 in the map by 2007?

6. The relation $T = 190\left(\frac{1}{2}\right)^{\frac{t}{10}}$ can be used to determine the length of time, t, in hours, that milk of a certain fat content will remain fresh. T is the storage temperature, in degrees Celsius.

a) What is the freshness half-life of milk?

b) Graph the relation.

c) How long will milk keep fresh at 22°C? at 4°C?

7. The remaining concentration of a particular drug in a person's bloodstream is modelled by the relation $C = C_0\left(\frac{1}{2}\right)^{\frac{t}{4}}$, where C is the remaining concentration of drug in the bloodstream in milligrams per millilitre of blood, C_0 is the initial concentration, and t is the time, in hours, that the drug is in the bloodstream.

a) What is the half-life of this drug?

b) A nurse gave a patient this drug. The concentration was 40 mg/mL, at 10:15 A.M. What will the concentration at

i) 3:15 P.M.? **ii)** 10:00 P.M.?

c) A second dose of the drug needs to be given to this patient when the concentration of drug in the bloodstream is down to 0.5 mg/mL. Estimate after how many hours this would occur.

Literacy Connect

8. The directions for taking a medication are shown.

ORAL DOSE:	ADULTS: 2 caplets every six hours
WARNING:	It is hazardous to exceed the recommended dose unless supervised by a physician.

When you take medication, the amount of drug in your bloodstream rises to a maximum, then decreases over time. The time it takes for half the drug to leave your bloodstream is called the half-life of the medication.

a) How do you think the half-life of a medication relates to the ORAL DOSE instructions that state "every six hours"?

b) Use the words *half-life* and *bloodstream* to explain how someone may receive an **overdose** if more medication is taken before the six-hour recommended period. To learn more about half-life, go to *www.mcgrawhill.ca/links/foundations11* and follow the links.

Chapter Problem

9. The relation $I = 10^{-12} \times 10^{\frac{dB}{10}}$ is used to calculate the intensity of sound, I, relative to the threshold of human hearing (0 dB), where dB represents the sound being compared. Copy and complete the table.

Sound Source	Intensity Level (dB)	Relative Intensity
Mosquito buzzing	40	$10^{-12} \times 10^{\frac{40}{10}} = 10^{-8} = 0.000\ 000\ 01$
Rainfall	50	
Quiet alarm clock	65	
Loud alarm clock	80	
Average factory	90	
Large orchestra	98	
Car stereo	125	

10. A general relation between speed and collision rates is "1 km faster results in a 3% increase in the collision rate." On a specific stretch of road, the collision rate is 0.534 collisions per million vehicle kilometres, when vehicles travel at an average speed of 80 km/h.

Literacy Connect

a) What is meant by "collisions per million vehicle kilometres"?

b) The relation that represents the collision rate for this stretch of road is $R = 0.534(1.03)^s$, where R is the collision rate, in collisions per million vehicle kilometres, and s is the average vehicle speed. What would be the collision rate if the average speed increases to

i) 90 km/h? **ii)** 120 km/h?

11. Research the growth of the Consumer Price Index since 1914. Go to *www.mcgrawhill.ca/links/foundations11* and follow the links.

a) Determine the cost of a $10 item in 1914 in subsequent years.

b) Enter the data into a spreadsheet or a graphing calculator and find the curve of best fit to model the data.

c) Use your model to predict how much an item costing $10 in 1914 will cost in 2010 and in 2020.

12. Research the world population from 1000 B.C.E. to current times. Go to *www.mcgrawhill.ca/links/foundations11* and follow the links.

a) Enter the data into a spreadsheet or a graphing calculator and find the curve of best fit to model the data.

b) Use your model to predict the world's population today. How accurate is your model? Explain any differences.

c) Use your model to predict the world's population in 2050.

13. In nuclear medicine, very small quantities of a radioactive material are injected into patients. The material is traced so a medical diagnosis can be made. On the label for the material, the supplier indicates the level of radioactivity at a specific date and time. A relation used by the doctor or technician is $A = A_0 \times \left(\frac{1}{2}\right)^{\frac{t}{h}}$, where A is the radioactivity after time t, A_0 is the radioactivity at a specific moment, and h is the half-life of the material. The unit of measure of radioactivity is the Bequerel (Bq), or disintegration per second.

a) Iodine-131 has a half-life of 8.065 days. The label states that the radioactivity level was 370 MBq (10^6 Bq) at 12:00 P.M. on March 4. What will be the radioactivity level at 12:00 P.M. on March 7? March 16?

b) Technetium-99 has a half-life of 6.007 h. The label states that the radioactivity level was 284 MBq at 8:20 A.M. on July 7. What will be the radioactivity level at 9:30 P.M. on the same day?

Extend C

14. The fox population of a national park was 325 foxes 15 years ago. Today, it is 650 foxes. Assuming the population has experienced exponential growth, write a relation representing the size of the fox population in the park. Use your relation to project the fox population in 20 years.

7 Review

7.1 Exponent Rules, pages 356–364

1. Write as a single power, then evaluate.

a) $6^2 \times 6^3$ **b)** $(-2)^2 \times (-2)^4$

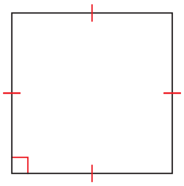
c) $\frac{5^{10}}{5^7}$ **d)** $\left(\frac{1}{3}\right)^3 \times \left(\frac{1}{3}\right)^3$

e) $(10^4)^2$ **f)** $[(-7)^2]^2$

g) $\frac{3^8}{3^5}$ **h)** $\left(-\frac{1}{2}\right)^2 \times \left(-\frac{1}{2}\right)^3$

2. Calculate the area of the square.

$A = s^2$



7.2 Zero and Negative Exponents, pages 365–371

3. Evaluate. Express your answers as whole numbers or fractions.

a) 7^0 **b)** 5^{-1} **c)** 8^{-3}

d) $\left(\frac{1}{50}\right)^0$ **e)** $\left(\frac{2}{3}\right)^{-2}$ **f)** $4^{-2} \times 4^5$

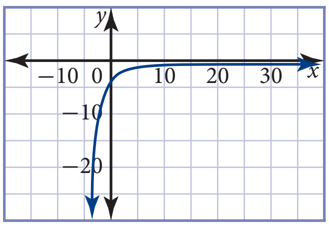
g) $\frac{7^2}{7^3}$ **h)** $[(-3)^2]^{-1}$ **i)** $\frac{1}{2^{-3}}$

j) $\frac{1}{5^{-1}}$

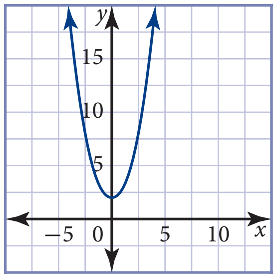
7.3 Investigate Exponential Relationships, pages 372–381

4. Which graph represents an exponential relationship? Explain.

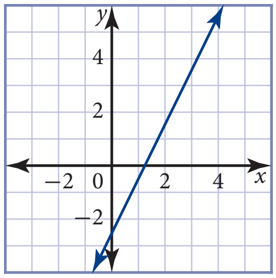
a)



b)



c)



5. A certain type of bacteria doubles every 8 h. A culture begins with 30 000 bacteria. How many bacteria are there after

a) 8 h? **b)** 16 h? **c)** 4 days?

7.4 Exponential Relations, pages 382–394

6. Compare the graphs of $y = 3^x$, $y = 3x^2$, and $y = 3x$, for values of $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$. How are they similar? different?

7. The measure of the acidity of a solution is called its pH. In wells and swimming pools, the pH level of water needs to be checked regularly for the level of hydrogen. The relation $H = \left(\frac{1}{10}\right)^P$ gives an indication of the concentration of hydrogen ions, H, in moles per litre (mol/L), where P represents the pH.

a) Plot a graph of this relation.

b) Water with a pH of less than 7.0 is acidic. What is the hydrogen concentration for a pH of 7.0?

c) Water in a swimming pool needs to be kept at a pH between 7.0 and 7.6. What is the equivalent range of hydrogen concentration?

d) Rain water has a pH of 5.6. Due to sulphur pollution, acid rain has a pH of less than 5.0. Compare the concentrations of hydrogen in rain and acid rain.

7.5 Modelling Exponential Growth and Decay, pages 395–405

8. A town's raccoon population is growing exponentially. The expected population can be estimated using the relation $P = 1250(1.013)^n$, where P is the population and n is the number of years.

a) Use technology to plot a graph of this relation.

b) What is the current raccoon population?

c) What is the expected population in 5 years?

9. The amplitude of a pendulum over a 60-s period is shown in the table.

Time (s)	0	10	20	30	40	50	60
Amplitude (cm)	80.0	40.0	20.0	10.0	5.0	2.5	1.25

a) Use technology to make a scatter plot.

b) Do the results of this experiment demonstrate exponential growth or decay? Explain.

c) What is the rate of growth or decay, in amplitude?

d) How long would it take for the swings to become unnoticeable (less than 0.2 cm)?

7.6 Solve Problems Involving Exponential Growth and Decay, pages 406–413

10. The remaining mass of a drug in a person's bloodstream is modelled by $M = 500\left(\frac{1}{2}\right)^{\frac{t}{2}}$, where M is the remaining mass in milligrams, and t is the time, in hours, that the drug is in the bloodstream.

a) What is the half-life of the drug?

b) What was the dosage of the drug?

c) What will be the concentration of the drug in the bloodstream

i) after 2 h?

ii) after 6 h?

11. From 1994 to 2004, average personal incomes grew in Canada according to the relation $I = I_0(1.041)^n$, where I is the resulting income, I_0 is the initial income, and n is the number of years of growth.

a) If a person's income was $34 000 in 1994, what would it be in 2004?

b) If a person's income was $50 000 in 1996, what would it be in 2003?

c) What was the average yearly rate of growth from 1994 to 2004?

7 Practice Test

1. True or false?
   a) Linear growth shows increases by a constant amount each time period.
   b) Exponential growth shows increases by a constant factor each time period.
   c) Exponential decay shows decreases by a fixed amount each time period.
   d) Quadratic growth is confirmed by unequal first differences and equal second differences.

2. Evaluate. Write your answers as integers or fractions.
   a) $3^3 \times 3^2$
   b) $\frac{9^7}{9^5}$
   c) $(2^3)^2$
   d) 6^0
   e) 7^{-2}
   f) $\left(\frac{1}{5}\right)^{-3}$
   g) $4^{12} \times 4^{-3} \times 4^{-9}$
   h) $\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^{-1}$

3. i) Sketch a graph of each relation.
   ii) Classify each as exponential growth, exponential decay, or neither. Justify your response.
   a) $y = \left(\frac{1}{4}\right)^x$
   b) $y = 6x^2$
   c) $y = 5^x$
   d) $y = 3(0.5)^{\frac{x}{4}}$

4. Topsoil is commonly measured in cubic yards (yd^3) and cubic feet (ft^3). A medium-sized dump truck can hold about 9 yd^3 of topsoil. A wheelbarrow can hold about 3 ft^3 of topsoil.
   a) How many cubic feet are in a cubic yard? (1 yd = 3 ft) Express your answer as a power.
   b) Express 9 yd^3 in cubic feet as a power with base 3.
   c) A family ordered 9 yd^3 of topsoil to landscape their yard. How many trips with a wheelbarrow will they need to make in order to move all the topsoil that was delivered?

5. You have investigated graphs of exponential relations of the form $y = b^x$ for $b > 0$. Explain why graphs of this form remain above the x-axis.

6. In 1878, moose were first introduced in Newfoundland with a single bull and a single cow. Today, there are approximately 150 000 moose in Newfoundland.
   a) On the same set of axes, draw a linear, a quadratic, and an exponential graph to represent growth of the moose population.
   b) Which type of growth is most likely? Explain.

Chapter Problem Wrap-Up

The table shows the recommended maximum continuous exposure times to loud sounds.

Sound Intensity	Recommended Maximum Continuous Exposure Time
85 dB	8.0 h
88 dB	4.0 h
91 dB	2 h
94 dB	1.0 h
97 dB	30.0 min
100 dB	15.0 min
103 dB	7.5 min
106 dB	3.75 min
109 dB	1.875 min
112 dB	0.9375 min
115 dB	0.46875 min

a) Consider your answers to the chapter problems in the previous sections. Explain why the relationship between sound intensity and exposure time is exponential.

b) Write a paragraph describing how continued exposure to loud sounds could affect hearing loss. Use the words "exponential relation" and "continuous exposure" in your answer.

7. Atmospheric pressure depends on the altitude above sea level. Altitude is measured in kilopascals (kPa).

Altitude (km)	Atmospheric Pressure (kPa)
0	101.3
1	89.4
2	78.9
3	69.7
4	61.5
5	54.3
6	47.9
7	42.3
8	37.3
9	32.9
10	29.1

a) Use graphing technology to create a scatter plot of the data.

b) Find an equation that models the data using exponential regression.

c) What is the atmospheric pressure at sea level?

d) At how many metres above sea level will a mountain climber experience atmospheric pressure of 89 kPa?

e) What is the atmospheric pressure 4250 m above sea level?