5 Quadratic Relations II

Many fountains are more than just decorative; they are attractions unto themselves. The streams of water follow a path in the shape of a parabola. Understanding these paths allows designers to combine several fountains to create beautiful patterns.

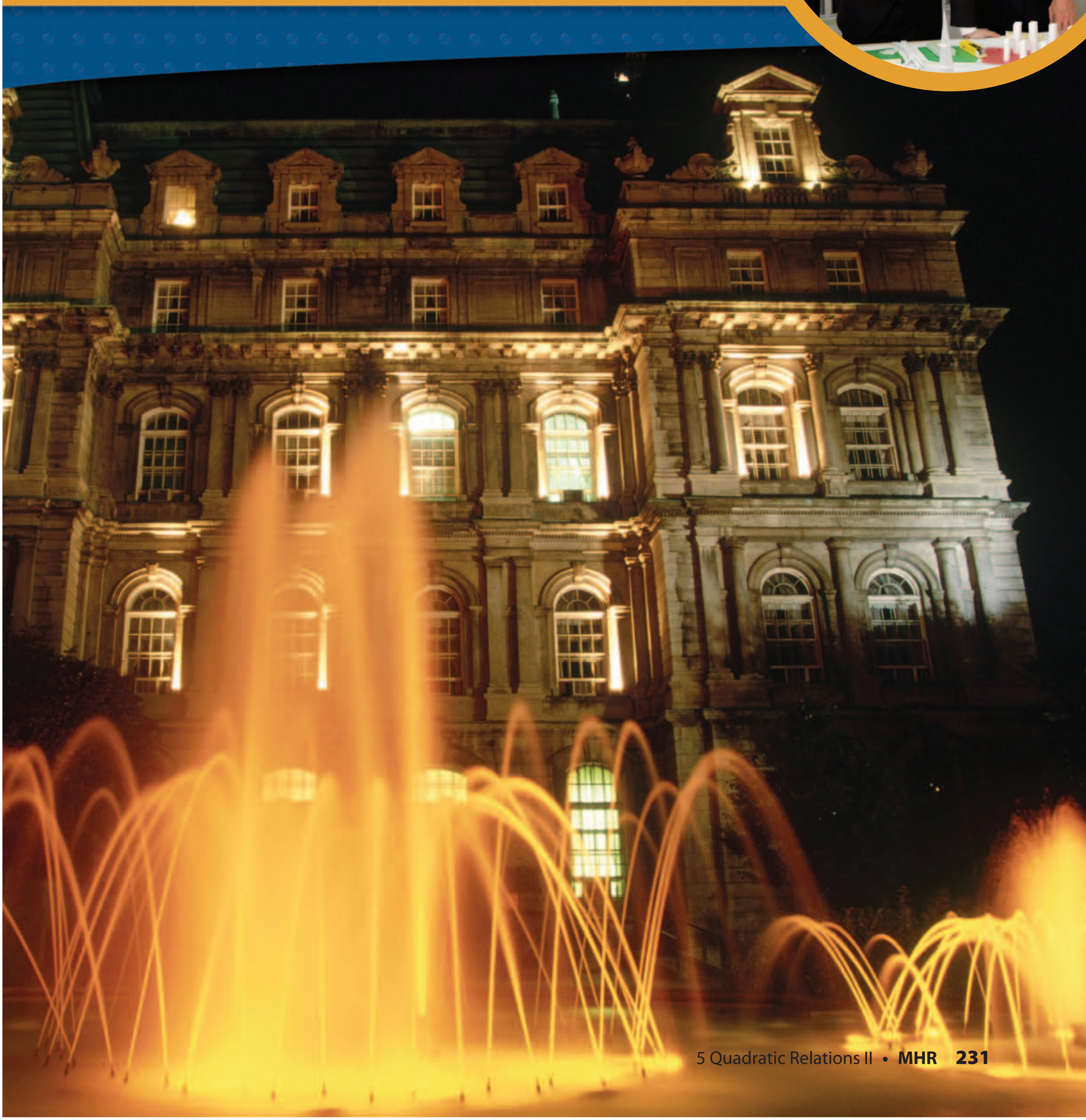
In this chapter, you will

- expand and simplify quadratic expressions in one variable involving multiplying binomials or squaring a binomial, using a variety of tools
- express the equation of a quadratic relation in the standard form $y = ax^2 + bx + c$, given the vertex form $y = a(x - h)^2 + k$, and verify, using graphing technology, that these forms are equivalent representations
- factor trinomials of the form $ax^2 + bx + c$, where $a = 1$ or where a is the common factor, by various methods
- determine, through investigation, and describe the connection between the factors of a quadratic expression and the x-intercepts of the graph of the corresponding quadratic relation
- solve problems, using an appropriate strategy, given equations of quadratic relations, including those that arise from real-world applications

Key Terms

axis of symmetry	perfect square trinomial
difference of squares	standard form
intercept form	zeros

Yao attended a two-year program at Mohawk College to become an architectural technician. In his job, Yao helps architects prepare blueprints and build models of proposed buildings. Mathematics is the key to successful construction of the architect's design. Mathematical shapes, such as parabolas, make the finished structure a work of art.



Prerequisite Skills

Polynomials

1. Simplify.

a) $6(3x)$ **b)** $-9(-15x)$

c) $11(-8x)$ **d)** $1.5(3x)$

2. Simplify.

a) $4x^2 - 3x + 9x^2 + 7x$

b) $3x + 2 - 5x + 15$

c) $10x^2 - 12x - 7x + 9$

d) $5x^2 - 3x + 5 - 7x^2 + 4x - 10$

3. Expand and simplify.

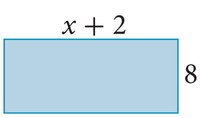
a) $4(x + 16)$ **b)** $3x(17 + 2x)$

c) $-7x(12x - 3)$ **d)** $10x(4x - 5)$

Area

4. Use algebra tiles to model each rectangle. Then, find an expression, in simplified form, for the area.

a)



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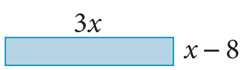


Draw and Interpret Graphs

5. Graph each linear relation.

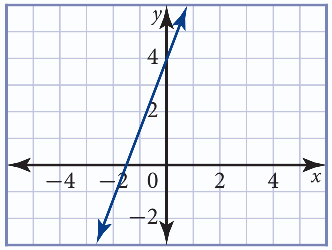
a) $y = x + 4$ **b)** $x + 4y - 3 = 0$

c) $y = -\frac{1}{4}x + 3$ **d)** $x - y + 4 = 0$

6. Which of the equations from question 5 represent the same relation?

7. Find the x- and y-intercepts of each relation.

a)



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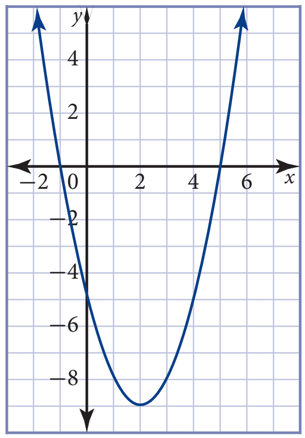

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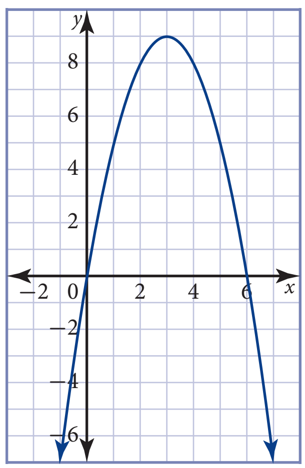


Algebraic Expressions

8. Use algebra tiles to model each expression.

a) $x^2 - 6x$ **b)** $x^2 - x - 2$

c) $3x^2 + 5x - 2$ **d)** $-2x^2 - 7x + 15$

9. Refer to question 8. Evaluate each expression for $x = -4$.

Number Skills

10. List the factors of each number.

a) 24 **b)** 81

c) 30 **d)** −18

11. Find two integers with each product and sum.

	Product	Sum
a)	21	10
b)	12	8
c)	20	12
d)	32	18
e)	50	27
f)	−20	1
g)	−64	0
h)	−64	−12

Solve Equations

12. Solve for x.

a) $3x = 15$ **b)** $17 = x + 4$

c) $x - 15 = 22$ **d)** $-5x = 65$

e) $4x - 7 = 21$ **f)** $-9x + 22 = -50$

g) $5x + 15 = 2x$ **h)** $-9x = 6x + 30$

Factor Polynomials

13. Find the greatest common factor, then factor each expression.

a) $3x + 9$ **b)** $5x + 20$

c) $7x - 35$ **d)** $-8x - 48$

e) $x^2 - 4x$ **f)** $4x^2 + 24x$

g) $-15x^2 + 27x$ **h)** $20x^2 - 55$

14. Factor each trinomial.

a) $x^2 + 3x + 2$ **b)** $x^2 + x - 6$

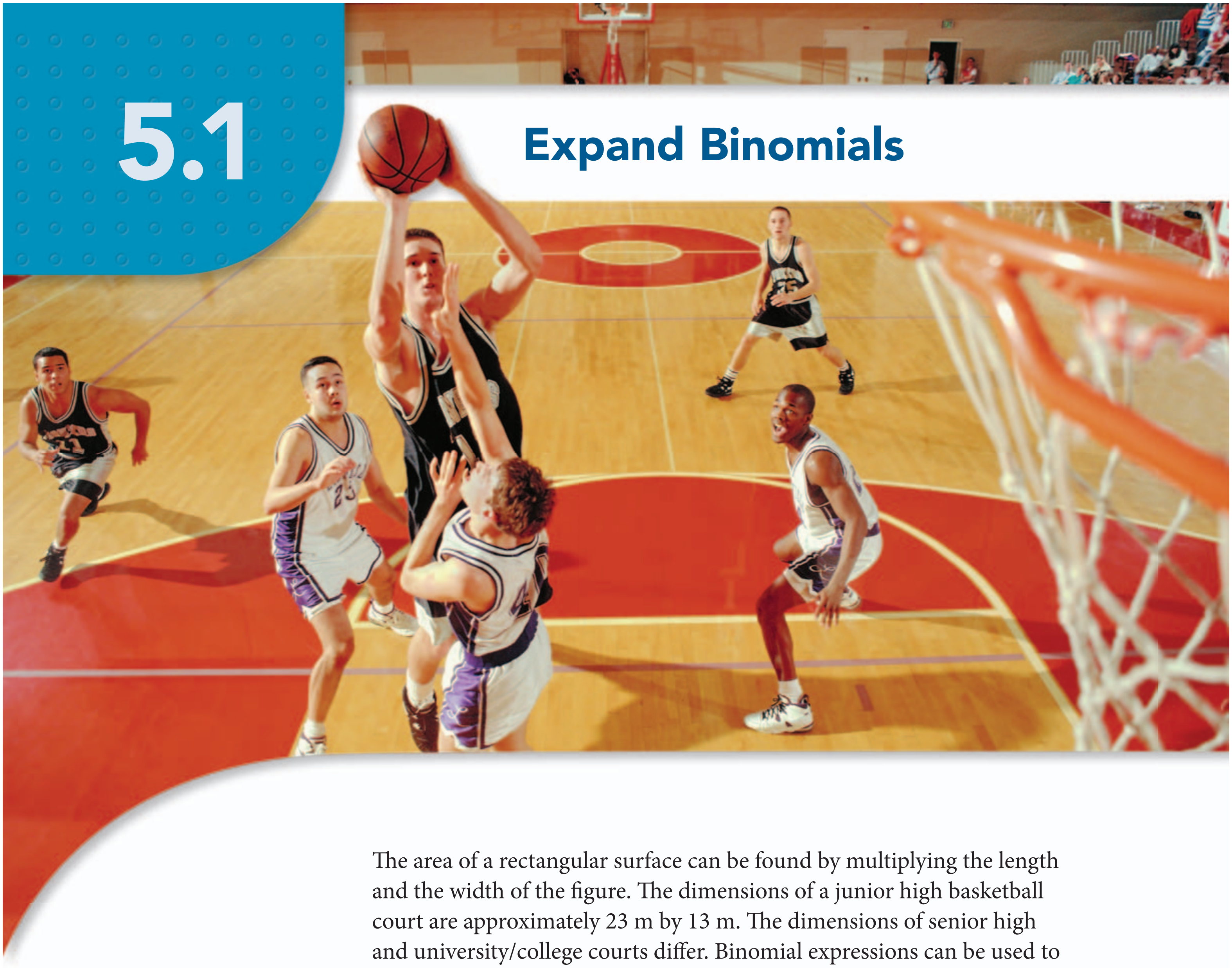
c) $x^2 - 8x + 12$ **d)** $x^2 + 9x + 14$

e) $x^2 - 3x - 10$ **f)** $x^2 - 2x + 1$

Chapter Problem

You have been hired to design a fountain that will be the main attraction for a new park. What should you consider when designing the fountain? How can you use your understanding of quadratic relations to make an attractive design?

5.1 Expand Binomials



The area of a rectangular surface can be found by multiplying the length and the width of the figure. The dimensions of a junior high basketball court are approximately 23 m by 13 m. The dimensions of senior high and university/college courts differ. Binomial expressions can be used to represent the dimensions of any basketball court.

The product of two numbers can be rewritten as the product of two sums. For example:

$$\begin{aligned}42 \times 64 &= (40 + 2) \times (60 + 4)\\ &= 40 \times 60 + 40 \times 4 + 2 \times 60 + 2 \times 4\\ &= 2400 + 160 + 120 + 8\\ &= 2688\end{aligned}$$

The product of two numbers can also be rewritten as the product of a sum and a difference or the product of two differences. For example:

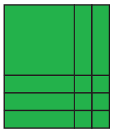
$$\begin{aligned}28 \times 75 &= (30 - 2) \times (70 + 5)\\ &= 30 \times 70 + 30 \times 5 - 2 \times 70 - 2 \times 5\\ &= 2100 + 150 - 140 - 10\\ &= 2100\end{aligned}$$

In this section, you will apply this method to multiply algebraic expressions.

Investigate

Model the Product of Binomials

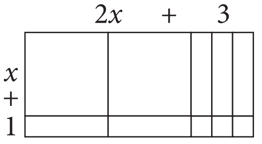
1. The diagram shows a rectangle made of algebra tiles.



a) Write an expression to represent each of the dimensions of the rectangle.

b) Write an expression to represent the area of the rectangle.

2. Copy this diagram.



a) Write an expression to represent the area of each small rectangle.

b) Write an expression to represent the area of the large rectangle.

3. **Reflect** Compare the expression that represents the area of the large rectangle to the expressions that represent its dimensions. How are they related? Explain.

Example 1

Multiply Two Binomials

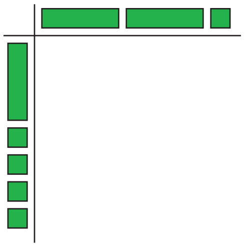
Expand and simplify.

$(2x + 1)(x + 4)$

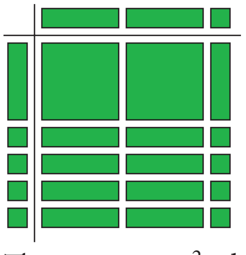
Solution

Method 1: Use Algebra Tiles

Draw a multiplication frame. Use algebra tiles to model each binomial.



Fill in the rectangle using algebra tiles.



There are two x^2-tiles, nine x-tiles, and four unit tiles.

So, $(2x + 1)(x + 4) = 2x^2 + 9x + 4$.

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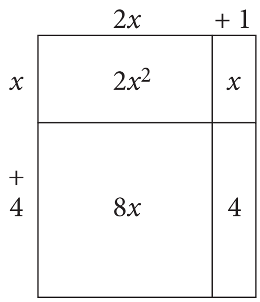
Method 2: Use an Area Model

Draw a large rectangle with length $2x + 1$ and width $x + 4$. Divide the large rectangle into smaller rectangles. Find the area of each small rectangle.



Multiply the length and the width:
- $2x \times x = 2x^2$
- $1 \times x = x$
- $2x \times 4 = 8x$
- $1 \times 4 = 4$

The sum of the areas of the small rectangles gives the area of the large rectangle.

$$\begin{aligned}(2x + 1)(x + 4) &= 2x^2 + x + 8x + 4\\ &= 2x^2 + 9x + 4\end{aligned}$$

Method 3: Use the Distributive Property

$$\begin{aligned}(2x + 1)(x + 4) &= \boxed{(2x + 1)}(x + 4)\\ &= (2x + 1)(x) + (2x + 1)(4)\\ &= (2x + 1)(x) + (2x + 1)(4)\\ &= 2x^2 + x + 8x + 4\\ &= 2x^2 + 9x + 4\end{aligned}$$

Math Connect

This method is often referred to by the acronym **FOIL**. Multiply the **F**irst terms, the **O**utside terms, the **I**nside terms, and the **L**ast terms.

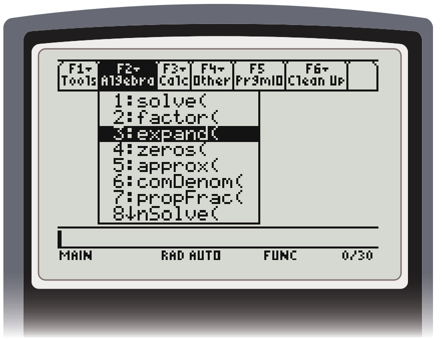
Method 4: Use a Multiplication Pattern

When multiplying polynomials, each term in one binomial must be multiplied by each term in the other binomial. Think of it as two pairs of people who meet at a party. Each person in one pair will shake hands with each person in the other pair.

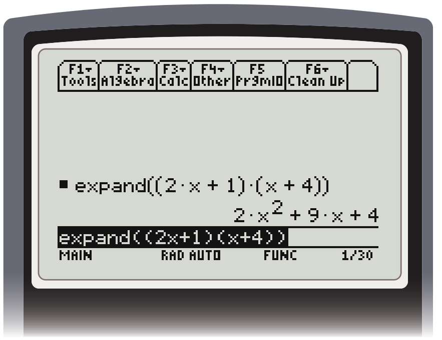
$$\begin{aligned}(2x + 1)(x + 4) &= (2x + 1)(x + 4)\\ &= (2x)(x) + (2x)(4) + (1)(x) + (1)(4)\\ &= 2x^2 + 8x + x + 4\\ &= 2x^2 + 9x + 4\end{aligned}$$

Method 5: Use a Computer Algebra System (CAS)

From the home screen, press [F2] **3:expand(**.



Press [(] 2 [x] [+] 1 [)] [(] [x] [+] 4 [)] [)] [ENTER].



$(2x + 1)(x + 4) = 2x^2 + 9x + 4$

Example 2

Find the Product of Two Binomials

Find each product, then simplify.

a) $(2x - 7)(4x + 9)$

b) $(3x - 5)(3x + 5)$

Solution

a) $(2x - 7)(4x + 9) = (2x - 7)(4x + 9)$

$$= 8x^2 + 18x - 28x - 63$$
$$= 8x^2 - 10x - 63$$

b) $(3x - 5)(3x + 5) = (3x - 5)(3x + 5)$

$$= 9x^2 + 15x - 15x - 25$$
$$= 9x^2 - 25$$

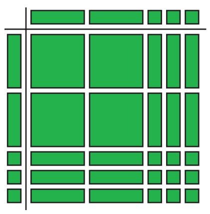
Example 3

Square a Binomial

Expand and simplify.
$(2x + 3)^2$

Solution

$$(2x + 3)^2 = (2x + 3)(2x + 3)$$
$$= 4x^2 + 6x + 6x + 9$$
$$= 4x^2 + 12x + 9$$



Check: Use algebra tiles.
When you use algebra tiles to square a binomial, the tiles that represent the product form a square.

Key Concepts

- To find the product of two binomials, each term in one binomial is multiplied by each term in the other binomial.

Discuss the Concepts

D1. Which method do you prefer to use when expanding binomials? Give reasons for your answer.

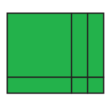
D2. How is expanding a binomial using an area model related to expanding a binomial using a multiplication pattern? Explain.

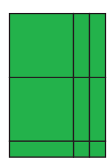
Practise A

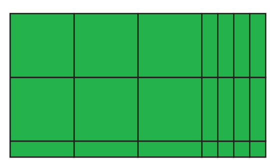
For help with questions 1 to 6, refer to Examples 1 and 2.

1. Write an expression for each dimension of each large rectangle.

a) 

b) 

c) 

d) 

2. For each large rectangle in question 1, write a simplified algebraic expression to represent its area.

3. Expand and simplify.

a) $x(x + 8)$
b) $(x + 7)(x + 1)$
c) $(x + 3)(x + 4)$
d) $(2x + 1)(x + 3)$
e) $(4x + 5)(6x + 2)$
f) $(3x + 2)(3x + 2)$

4. Expand and simplify.

a) $(3x + 5)(4x + 7)$
b) $(6x - 11)(x + 4)$
c) $(9x - 6)(2x + 10)$
d) $(7x + 2)(5x - 8)$
e) $(3 - 8x)(3 + 8x)$
f) $(4x + 9)(4x + 9)$

5. Expand and simplify.

a) $(2x + 3)(x + 10)$
b) $(3x + 10)(x + 5)$
c) $(x - 12)(3x + 11)$
d) $(5 - 10x)(15 + 2x)$
e) $(4x - 9)(x - 15)$
f) $(16x + 9)(16x + 9)$

6. Expand and simplify.

a) $(x - 5)(x + 5)$
b) $(x - 10)(x + 10)$
c) $(3x + 7)(3x - 7)$
d) $(8x - 5)(8x + 5)$
e) $(7x - 7)(7x + 7)$
f) $(12x + 9)(12x - 9)$

For help with question 7, refer to Example 3.

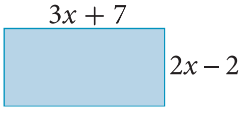
7. Expand and simplify.

a) $(x + 6)(x + 6)$
b) $(x - 8)(x - 8)$
c) $(4x + 15)^2$
d) $(9x - 2)^2$
e) $(5x - 3)^2$
f) $(6x + 12)^2$

Apply B

8. Refer to your answers to questions 6 and 7. Compare the factors with the products. Describe any patterns you see.

9. a) Write a simplified expression for the area of each rectangle.

i)


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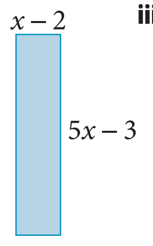

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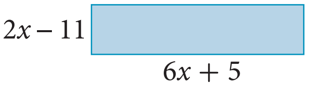


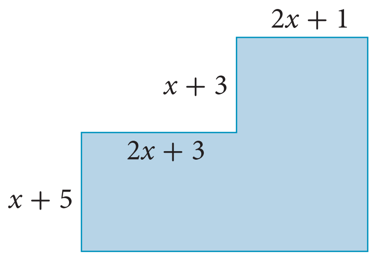
b) Determine each area if $x = 12$ cm.

10. Dania's yard has dimensions $s + 6$ by $2s - 5$.

a) Write an expression, in simplified form, for the area of Dania's yard.

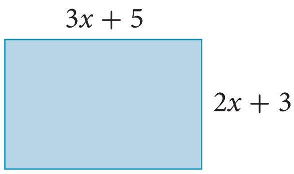
b) If $s = 10$ m, find the area of Dania's yard.

11. Write an expression, in simplified form, for the area of this shape.



Chapter Problem

12. The city planners have not confirmed the exact size of the fountain but you know that the base of the fountain will be rectangular. You have determined expressions for the dimensions of the base, in metres, as shown.



a) Write an expression, in simplified form, to represent the area of the base of the fountain.

b) When $x = 0$, the base of the fountain will be 5 m by 3 m, the minimum dimensions requested by the planners. The planners do not want the fountain to be too large, so you have designed it to have a maximum size when $x = 3$. How much greater in area is the base of the largest fountain than the base of the smallest fountain?

c) The projected cost for the base, including labour and materials, is $\$900/m^2$. What are the projected costs for the bases of the smallest and largest fountains?

Literacy Connect

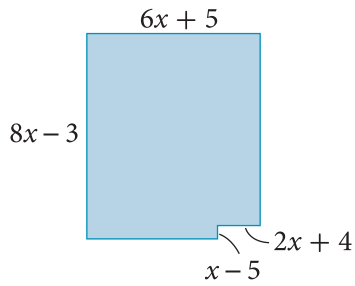
13. A rectangle in which the ratio of the length to the width is approximately 1.618:1 is called a golden rectangle. The ancient Greeks frequently used this proportion in their architecture and art.

a) Refer to question 12. What is the value of x if the base of the fountain is a golden rectangle?

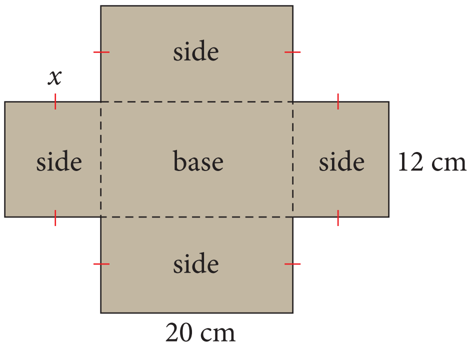
b) This ratio is more formally called phi. Research the accurate value of phi. Determine where else this ratio occurs in nature, architecture, and art. Write a report of your findings.

Extend C

14. Write an expression, in simplified form, for the area of this shape.



15. A box with a rectangular base and no lid can be created from a cardboard template as shown. The height of the box, x, is variable.



a) Find an expression for the area of the cardboard.

b) Calculate the total area of the cardboard for boxes with heights of 3 cm, 5 cm, and 10 cm.

c) If the cardboard costs 5¢/100 cm^2, how much will each box in part b) cost?

16. When multiplying two polynomials, each term in the first polynomial is multiplied by each term in the second polynomial. Expand and simplify.

a) $(3x + 2)(x^2 + 4x + 9)$

b) $(2x - 5)(7x^2 - 2x + 8)$

c) $(x^2 + 10x + 1)(x^2 - 3x + 11)$

17. Write each expression as the product of two binomials. Use the patterns that you found in question 8 to help you.

a) $x^2 + 10x + 25$

b) $x^2 - 18x + 81$

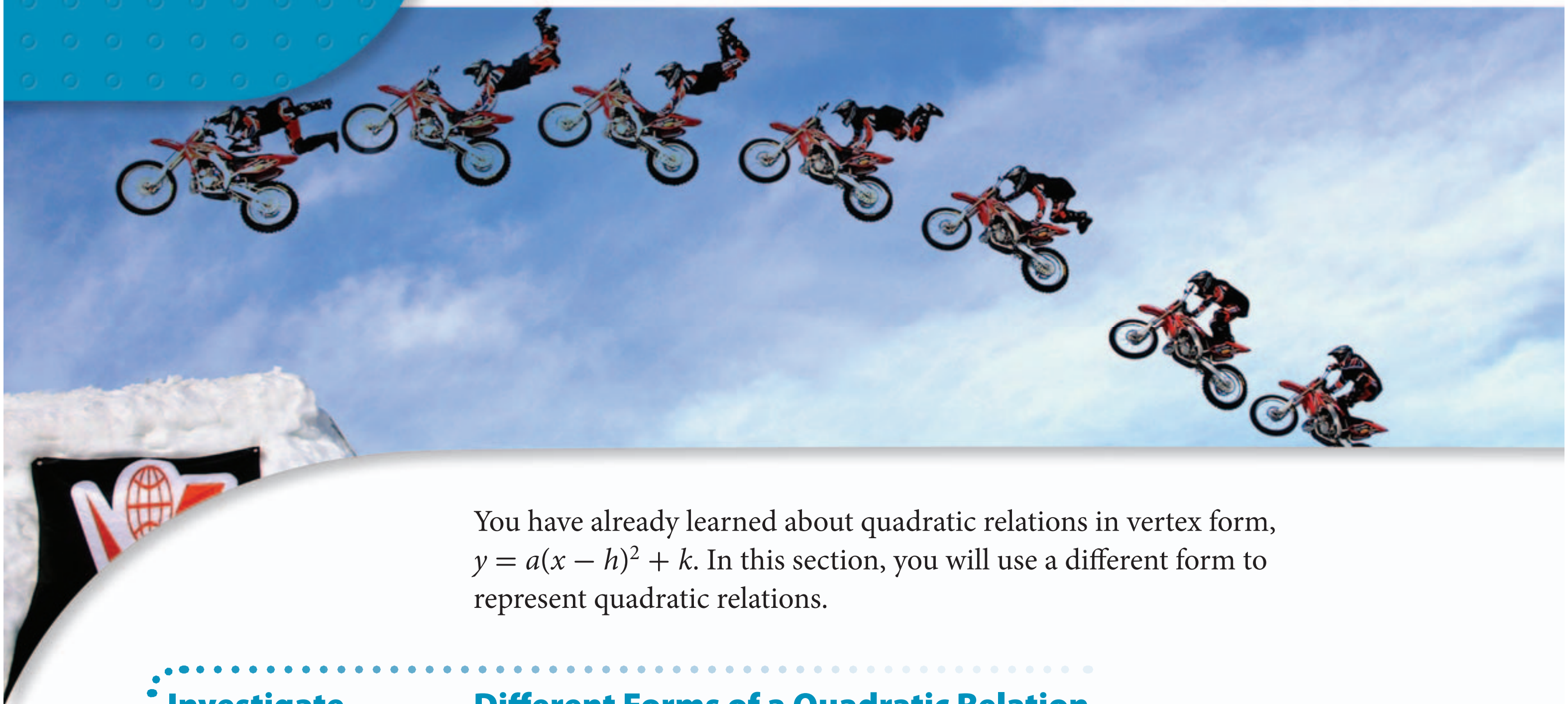
c) $x^2 + 24x + 144$

d) $x^2 - 36$

e) $x^2 - 64$

f) $x^2 - 121$

5.2 Change Quadratic Relations From Vertex Form to Standard Form



You have already learned about quadratic relations in vertex form, $y = a(x - h)^2 + k$. In this section, you will use a different form to represent quadratic relations.

Investigate

Tools

- graphing calculator

Different Forms of a Quadratic Relation

1. Graph each pair of relations in the same window. Use the standard viewing window. What do you notice?

   a) $y = (x - 2)^2 + 1$ $\qquad y = x^2 - 4x + 5$

   b) $y = (x + 5)^2 + 3$ $\qquad y = x^2 + 10x + 28$

   c) $y = -(x + 1)^2 - 4$ $\qquad y = -x^2 - 2x - 5$

   d) $y = 0.5(x - 4)^2 + 3$ $\qquad y = 0.5x^2 - 4x + 11$

2. Refer to question 1. Expand and simplify the first relation in each pair. How does the result compare to the second relation in each pair?

3. For each pair of relations in question 1, find the coordinates of the vertex and the y-intercept. Compare the coordinates of the vertex and the y-intercept to the relations. What do you notice?

4. The quadratic relation $y = ax^2 + bx + c$ is in **standard form**. The relation $y = a(x - h)^2 + k$ is in vertex form. Which relations in question 1 are written in standard form? Which are written in vertex form?

5. **Reflect** Without graphing, what do you know about a parabola given the relation in standard form? in vertex form? Use examples to explain.

standard form
- a quadratic relation of the form $y = ax^2 + bx + c$
- the constant, c, represents the y-intercept of the relation

Example 1

Write Quadratic Relations in Standard Form

Each relation is in vertex form. Write each relation in standard form.

a) $y = (x + 6)^2$

b) $y = -2(x - 7)^2$

c) $y = 3(x - 5)^2 - 8$

Solution

a)
$$\begin{aligned} y &= (x + 6)^2 \\ &= (x + 6)(x + 6) \\ &= x^2 + 6x + 6x + 36 \\ &= x^2 + 12x + 36 \end{aligned}$$

The relation $y = (x + 6)^2$ in standard form is $y = x^2 + 12x + 36$.

b)
$$\begin{aligned} y &= -2(x - 7)^2 \\ &= -2(x - 7)(x - 7) \\ &= -2(x^2 - 7x - 7x + 49) \\ &= -2(x^2 - 14x + 49) \\ &= -2x^2 + 28x - 98 \end{aligned}$$

The relation $y = -2(x - 7)^2$ in standard form is $y = -2x^2 + 28x - 98$.

c)
$$\begin{aligned} y &= 3(x - 5)^2 - 8 \\ &= 3(x - 5)(x - 5) - 8 \\ &= 3(x^2 - 5x - 5x + 25) - 8 \\ &= 3(x^2 - 10x + 25) - 8 \\ &= 3x^2 - 30x + 75 - 8 \\ &= 3x^2 - 30x + 67 \end{aligned}$$

The relation $y = 3(x - 5)^2 - 8$ in standard form is $y = 3x^2 - 30x + 67$.

Example 2

Projectile Motion

The path of a projectile can be modelled by the relation $y = -4.9t^2 + vt + h$, where t is the time, in seconds, since launching; y is the projectile's height, in metres; h is the projectile's initial height, in metres; and v is the projectile's initial velocity, in metres per second. Find the initial velocity and the initial height of a projectile that reaches a maximum height of 50 m after 3 s.

Solution

The maximum height of 50 m is reached after 3 s. So, the vertex is (3, 50).

Use the coordinates of the vertex and the given value of a, -4.9, to write the quadratic relation in vertex form. Then, convert the relation to standard form.

$$
\begin{aligned}
y &= -4.9(t - 3)^2 + 50 \\
&= -4.9(t - 3)(t - 3) + 50 \\
&= -4.9(t^2 - 3t - 3t + 9) + 50 \\
&= -4.9(t^2 - 6t + 9) + 50 \\
&= -4.9t^2 + 29.4t - 44.1 + 50 \\
&= -4.9t^2 + 29.4t + 5.9
\end{aligned}
$$

For a given quadratic relation, the vertex form and the standard form have the same *a*-value.

The initial velocity, v, is the coefficient of t. The projectile's initial velocity is 29.4 m/s.

The initial height, h, is the y-intercept. The projectile's initial height is 5.9 m.

Key Concepts

- A quadratic relation can be written in vertex form, $y = a(x - h)^2 + k$, or in standard form, $y = ax^2 + bx + c$.
- Expand and simplify the vertex form to write the quadratic relation in standard form.
- Given a quadratic relation in vertex form, $y = a(x - h)^2 + k$, the coordinates of the vertex are (h, k).
- Given a quadratic relation in standard form, $y = ax^2 + bx + c$, the y-intercept is c.

Discuss the Concepts

D1. A quadratic relation may be written in standard form or vertex form. What information does each form give about the parabola?

D2. Explain how you would convert the relation $y = -6(x - 10)^2 + 15$ to standard form.

Practise A

For help with questions 1 to 4, refer to Example 1.

1. Write each relation in standard form.

a) $y = (x + 6)^2$
b) $y = (x - 4)^2$
c) $y = (x - 15)^2$
d) $y = (x - 2)^2$
e) $y = (x + 9)^2$
f) $y = (x - 1)^2$

2. Write each relation in standard form.

a) $y = 3(x + 9)^2$
b) $y = -2(x + 7)^2$
c) $y = -8(x - 5)^2$
d) $y = 0.5(x + 2)^2$
e) $y = -0.25(x + 8)^2$
f) $y = 9.8(x - 3.2)^2$

3. Write each relation in standard form.

a) $y = (x - 8)^2 + 3$
b) $y = (x + 5)^2 + 10$
c) $y = (x + 1)^2 - 13$
d) $y = (x - 3)^2 + 1$
e) $y = (x + 6)^2 - 7$
f) $y = (x - 5)^2 - 3$

4. Write each relation in standard form.

a) $y = 5(x - 4)^2 + 12$
b) $y = -6(x + 9)^2 - 7$
c) $y = -2(x + 7)^2 - 10$
d) $y = -8(x - 5)^2 + 6$
e) $y = 2.4(x - 5.1)^2 + 3$
f) $y = -1.9(x + 2.7)^2 - 5.1$

For help with question 5, refer to the Investigate.

5. Graph each relation. Which relations are the same?

a) $y = 10(x - 7)^2 - 4$
b) $y = -2x^2 - 8x - 2$
c) $y = -2(x - 1)^2 + 5$
d) $y = -2(x + 2)^2 + 6$
e) $y = 10x^2 - 140x + 486$
f) $y = 10(x + 5)^2 + 3$
g) $y = -2x^2 + 4x + 3$
h) $y = 10x^2 + 100x + 253$

For help with questions 6 and 7, refer to Example 2.

6. For each quadratic relation, write an equation in standard form.

a) $a = 5$, vertex at $(1, 7)$
b) $y = -3x^2 + bx + c$, vertex at $(-5, 6)$
c) $a = -8$, maximum of 17 when $x = 10$
d) $y = 12x^2 + bx + c$, minimum of 3 when $x = -1$

7. Determine the y-intercept for each relation.

a) $y = 3(x + 12)^2 + 15$
b) $y = 10x^2 - 15x + 7$
c) $y = -7(x - 5)^2 - 6$
d) $y = 9x^2 - 20$
e) $y = 4x^2 + 5x - 1$
f) $y = 1.5(x - 2.4)^2 + 6.4$

Apply B

8. The strategy for long-distance rollerblading is to find the best speed that can be maintained for a long time. A racer's performance can be modelled by the quadratic relation $d = -2(v - 6)^2 + 50$, where d is the racer's maximum distance travelled, in kilometres, at a speed, v, in metres per second.
   a) Find the v-coordinate of the vertex. What does this value represent?
   b) Find the d-coordinate of the vertex. What does this value represent?
   c) Write the quadratic relation in standard form.
   d) Use technology to graph both forms of the relation. Verify the coordinates of the vertex.

Literacy Connect

9. Explain the advantages and disadvantages of each form of a quadratic relation.
   a) standard form, $y = ax^2 + bx + c$
   b) vertex form, $y = a(x - h)^2 + k$

10. The curve of a suspension cable on the Golden Gate Bridge in San Francisco, California, can be modelled by the quadratic relation $h = 0.000\,549x^2 + bx + c$, where h is the cable's height above the ground, and x is the horizontal distance from one tower, both in metres. The centre of the cable is 640 m from the tower and 227 m above the ground.
    a) Write the quadratic relation that models the curve of the cable in vertex form.
    b) Write the quadratic relation in standard form.
    c) At what height does a cable attach to a tower?
    d) Graph the relations from parts a) and b) on the same set of axes.

Chapter Problem

11. Water from the main fountain for your project must reach a maximum height of 20 m after 2 s. The water's path can be modelled by the quadratic relation $y = -4.9t^2 + vt + h$, where y is the water's height, in metres, h is the fountain's height above the ground, in metres, and v is the initial velocity of the water, in metres per second.
    a) Write the quadratic relation that models the path of the water in vertex form.
    b) Write the quadratic relation in standard form.
    c) What is the initial velocity of the water?
    d) What is the maximum height reached by the water?

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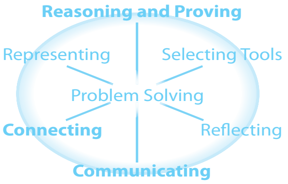


Achievement Check

12. The distributive property can be used to square whole numbers close to 50, without using a calculator. Consider these examples:

$$
\begin{aligned}
53^2 &= (53)(53) \\
&= (50 + 3)(50 + 3) \\
&= 2500 + 100n + n^2 \\
&= 2500 + 300 + 9 \\
&= 2809
\end{aligned}
\qquad
\begin{aligned}
46^2 &= (46)(46) \\
&= (50 - 4)(50 - 4) \\
&= 2500 + 100n + n^2 \\
&= 2500 - 400 + 16 \\
&= 2116
\end{aligned}
$$

a) Use the distributive property to calculate 56^2 and 48^2.

b) Explain how to calculate the square of a number that is x units away from 50. Write a statement similar to the examples above to illustrate your explanation.

c) Test this method using whole numbers close to 20. Does the pattern work for any value or only for values close to 50? Explain your reasoning.

Extend C

13. Step-Up is a new extreme sport in which a motorcycle rider uses a steep ramp to jump a high bar, as in the high jump or pole vault. The path of the rider can be modelled by the quadratic relation $y = -\frac{2106}{v^2}x^2 + 5.85x + h$, where y is the height of the rider above the ground, x is the rider's horizontal distance from the ramp, and h is the height of the ramp, all in metres, and v is the rider's initial velocity, in kilometres per hour.

a) If the rider's initial velocity is 36 km/h, determine the a-value for the quadratic relation.

b) A rider is 1.8 m from the ramp when the maximum height of 8.0 m is reached. Write this relation in vertex form.

c) What is the height of the ramp?

14. A quadratic relation can be represented by an equation of the form $y = a(x - s)(x - t)$.

a) Show that $y = 3(x - 1)^2 - 48$ represents the same quadratic relation as $y = 3(x + 3)(x - 5)$.

b) Graph both relations on the same set of axes.

c) Compare the relation $y = 3(x + 3)(x - 5)$ to the graph. What connections do you see between the graph and the relation?

5.3 Factor Trinomials of the Form $x^2 + bx + c$

In mathematics, opposite operations are used to undo operations. For addition, the opposite operation is subtraction. For multiplication, the opposite operation is division. For expanding, the opposite operation is factoring.

You can find the dimensions of a rectangular surface if you know its area. The surface area of a tennis court is approximately 264 m^2. If the width of the court is 11 m, what is the length?

Investigate

The Pattern in the Product of Two Binomials

1. Expand and simplify.

a) $(x + 6)(x + 1)$

b) $(x + 5)(x + 5)$

c) $(x + 2)(x + 3)$

d) $(x + 4)(x - 7)$

e) $(x - 1)(x + 4)$

f) $(x - 2)(x - 8)$

2. Refer to your answers to question 1.

a) Compare the constant terms in each pair of binomial factors to the constant term in the trinomial. What do you notice?

b) Compare the constant terms in each pair of binomial factors to the coefficient of x in the trinomial. What do you notice?

c) Compare the signs of the constant terms in each pair of binomial factors to the signs of the coefficient of x and the constant term in the trinomial. What do you notice?

3. Predict the coefficient of x and the constant term of the trinomial produced by each pair of binomial factors. Record your prediction. Then, expand to check your prediction.

a) $(x + 1)(x + 4)$

b) $(x + 6)(x + 2)$

c) $(x + 8)(x - 1)$

d) $(x - 3)(x - 5)$

e) $(x - 10)(x + 4)$

f) $(x - 5)(x - 1)$

4. Reflect For the trinomial $x^2 + 7x + 12$, how could you identify the constant terms in the binomial factors $(x + \blacksquare)(x + \blacksquare)$?

Example 1

Use Algebra Tiles to Factor a Trinomial

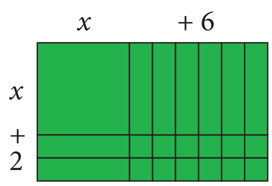
a) Factor $x^2 + 8x + 12$ using algebra tiles.

b) Check your answer by expanding, using the distributive property.

Solution

a) Use algebra tiles to model $x^2 + 8x + 12$.
Arrange the tiles to form a rectangle.

The dimensions of the rectangle represent the factors of the trinomial.



$$x^2 + 8x + 12 = (x + 6)(x + 2)$$

b) $$\begin{aligned}(x + 6)(x + 2) &= x^2 + 2x + 6x + 12\\ &= x^2 + 8x + 12\end{aligned}$$

By expanding, the original trinomial is obtained.

Example 2

Factor Trinomials

Find the binomial factors of each trinomial.

a) $x^2 + 15x + 36$

b) $x^2 + 7x - 18$

c) $x^2 - 10x + 25$

Solution

Find two numbers:

- whose product equals the constant term of the trinomial
- whose sum equals the coefficient of the x-term of the trinomial

a) $x^2 + 15x + 36$

List pairs of numbers whose product is 36.
Choose the pair of numbers whose sum is 15.

Product of 36		Sum
1	36	37
2	18	20
3	12	15
4	9	13
6	6	12

Since the product, 36, and the sum, 15, are both positive, the numbers are both positive.

The numbers are 3 and 12, so $x^2 + 15x + 36 = (x + 3)(x + 12)$.

b) $x^2 + 7x - 18$

List pairs of numbers whose product is -18.
Choose the pair of numbers whose sum is 7.

Product of -18		Sum
1	-18	-17
-1	18	17
2	-9	-7
-2	9	7
3	-6	-3
-3	6	3

Since the product, -18, is negative, the numbers have opposite signs.

The numbers are -2 and 9, so $x^2 + 7x - 18 = (x - 2)(x + 9)$.

c) $x^2 - 10x + 25$

List pairs of numbers whose product is 25.
Choose the pair of numbers whose sum is -10.

Product of 25		Sum
1	25	26
-1	-25	-26
5	5	10
-5	-5	-10

Since the product, 25, is positive and the sum, -10, is negative, both numbers are negative.

The numbers are -5 and -5.

$$\text{So, } x^2 - 10x + 25 = (x - 5)(x - 5)$$
$$= (x - 5)^2$$

The trinomial $x^2 - 10x + 25$ is an example of a **perfect square trinomial**.

perfect square trinomial
- a trinomial with identical binomial factors
- the result of squaring a binomial

Example 3

Factor $x^2 + bx$

Factor.

a) $x^2 + 5x$ **b)** $x^2 - 7x$

Solution

Method 1: Rewrite as a Polynomial of the Form $x^2 + bx + c$

a) $x^2 + 5x$ can be rewritten as $x^2 + 5x + 0$.

$$x^2 + 5x = x^2 + 5x + 0$$

Find pairs of numbers whose product is 0. For a product of 0, at least one factor must be 0
Choose the pair of numbers whose sum is 5.

$$= (x + 0)(x + 5)$$
$$= x(x + 5)$$

b) $x^2 - 7x = x^2 - 7x + 0$
$$= (x + 0)(x - 7)$$
$$= x(x - 7)$$

Method 2: Find the Greatest Common Factor (GCF)

a) The GCF of $x^2 + 5x$ is x.
So, $x^2 + 5x = x(x + 5)$

b) The GCF of $x^2 - 7x$ is x.
So, $x^2 - 7x = x(x - 7)$

Example 4

Factor a Difference of Squares

Factor. Describe any patterns you see in the answers.

a) $x^2 - 25$ **b)** $x^2 - 81$

Solution

a) $x^2 - 25$ can be rewritten as $x^2 + 0x - 25$.

$$\begin{aligned} x^2 - 25 &= x^2 + 0x - 25 \\ &= (x - 5)(x + 5) \end{aligned}$$

$-5 \times 5 = -25$, $-5 + 5 = 0$

b) $x^2 - 81$ can be rewritten as $x^2 + 0x - 81$.

$$\begin{aligned} x^2 - 81 &= x^2 + 0x - 81 \\ &= (x - 9)(x + 9) \end{aligned}$$

$-9 \times 9 = -81$, $-9 + 9 = 0$

In both answers, the binomial factors are almost the same, but with opposite signs.

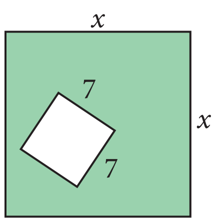
difference of squares
- a binomial of the form $x^2 - r^2$
- the factors of a difference of squares are $(x - r)(x + r)$

In Example 4, each polynomial is a difference of two perfect squares. The first term in each binomial factor is the square root of the first term in the polynomial. The second terms in the binomial factors are the positive and negative square roots of the second term of the polynomial. This type of polynomial is called a **difference of squares**.

Example 5

Difference of Areas

Find an expression, in factored form, for the shaded area of this figure.



Solution

The area of the large square is x^2. The area of the small square is 49 square units. The shaded area is the difference between the areas of the large square and the small square.

$$\begin{aligned} A &= x^2 - 49 \\ &= (x + 7)(x - 7) \end{aligned}$$

This is a difference of squares. The second terms of the binomial factors are the positive and negative square roots of 49.

Key Concepts

- Factoring is the opposite of expanding.
- To factor $x^2 + bx + c$, find two numbers whose product is equal to c and whose sum is equal to b.
- To factor $x^2 + bx$, rewrite as $x^2 + bx + 0$ or find the greatest common factor.
- A polynomial in the form $x^2 - r^2$ is a difference of squares. The factors are $(x + r)(x - r)$.
- Check the factors by expanding.

Discuss the Concepts

D1. Explain the steps needed to factor the expression $x^2 + 6x + 8$, using algebra tiles.

D2. Explain how to factor $x^2 + 6x + 8$ algebraically.

D3. What are the factors of $x^2 - 9$? Explain.

Practise A

For help with questions 1 to 3, refer to Example 2.

1. Find two numbers that have the given product and sum.

	Product	Sum
a)	25	10
b)	32	12
c)	24	−14
d)	36	−20
e)	−30	1
f)	−42	−11
g)	−50	23
h)	−64	0

2. Factor. Check your answers by expanding.

a) $x^2 + 15x + 36$

b) $x^2 + 8x + 16$

c) $x^2 + 12x + 20$

d) $x^2 + 13x + 40$

3. Factor each trinomial.

a) $x^2 - 13x + 22$

b) $x^2 - 14x + 49$

c) $x^2 - 11x + 28$

d) $x^2 - 20x + 100$

e) $x^2 + 14x - 32$

f) $x^2 + 13x - 48$

g) $x^2 - x - 20$

h) $x^2 - 18x - 63$

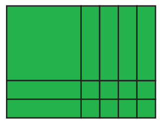
For help with questions 4 and 5, refer to Example 1.

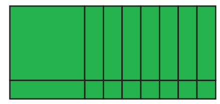
4. Model each expression using algebra tiles. Then, factor the expression.

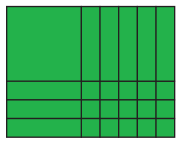
a) $x^2 + 3x + 2$ **b)** $x^2 + 7x + 6$

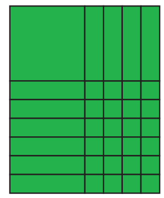
c) $x^2 + 8x + 12$ **d)** $x^2 + 6x + 8$

5. For each rectangle, write a trinomial expression for its area. Then, write an expression for each of its dimensions.

a)


b)


c)


d)


For help with question 6, refer to Example 3.

6. Factor.

a) $x^2 + 5x$ **b)** $x^2 + 22x$ **c)** $x^2 - 19x$

d) $x^2 - 15x$ **e)** $x^2 - 9.8x$ **f)** $x^2 + 33.5x$

For help with question 7, refer to Example 4.

7. Factor, then check by expanding.

a) $x^2 - 25$ **b)** $x^2 - 100$ **c)** $x^2 - 121$

d) $x^2 - 1$ **e)** $x^2 - 49$ **f)** $x^2 - 144$

8. Factor.

a) $x^2 + 25x$ **b)** $x^2 + 16x + 28$

c) $x^2 - x - 42$ **d)** $x^2 - 64$

e) $x^2 + 13x + 36$ **f)** $x^2 - 12x + 36$

g) $x^2 - 4$ **h)** $x^2 - 32x$

Apply B

9. Factor each trinomial, if possible.

a) $x^2 + 4x + 3$ **b)** $x^2 + 3x + 3$

c) $x^2 + 3x + 4$ **d)** $x^2 + 3x + 2$

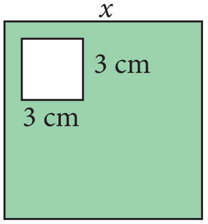
e) $x^2 - 4x + 3$ **f)** $x^2 + 2x - 3$

Literacy Connect

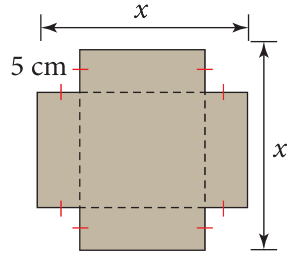
10. Refer to the trinomials in question 9 that you could not factor. Explain why you could not factor each of these trinomials.

11. Find an expression, in factored form, for the area of the shaded region of each figure.

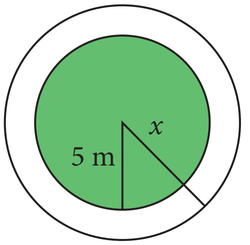
a)



b)



12. A circular garden with radius 5 m is surrounded by a walkway with radius x.



a) Write an expression for the total area of the garden and the walkway.

b) Write an expression for the area of the garden. Do not evaluate the expression.

c) Write an expression for the area of the walkway.

d) Factor the expression from part c).

Chapter Problem

13. City planners now wish to have a fountain with a square base. They want a 2-m square platform, which will remain dry, centred on the base. The area of the base not covered by the platform will have water on it.

a) The base of the fountain has side length x. Write an expression for the area that will have water on it.

b) The area of the base not covered by the platform will be tiled. The cost of the tile is $\$50/\text{m}^2$. How much will it cost to tile this area if the base of the fountain has side length 10 m?

14. The area of a rectangular garden can be represented by the expression $x^2 + 7x + 10$.

a) Find expressions for the length and width of the garden.

b) If the area is 40 m^2, find the length and width.

Extend C

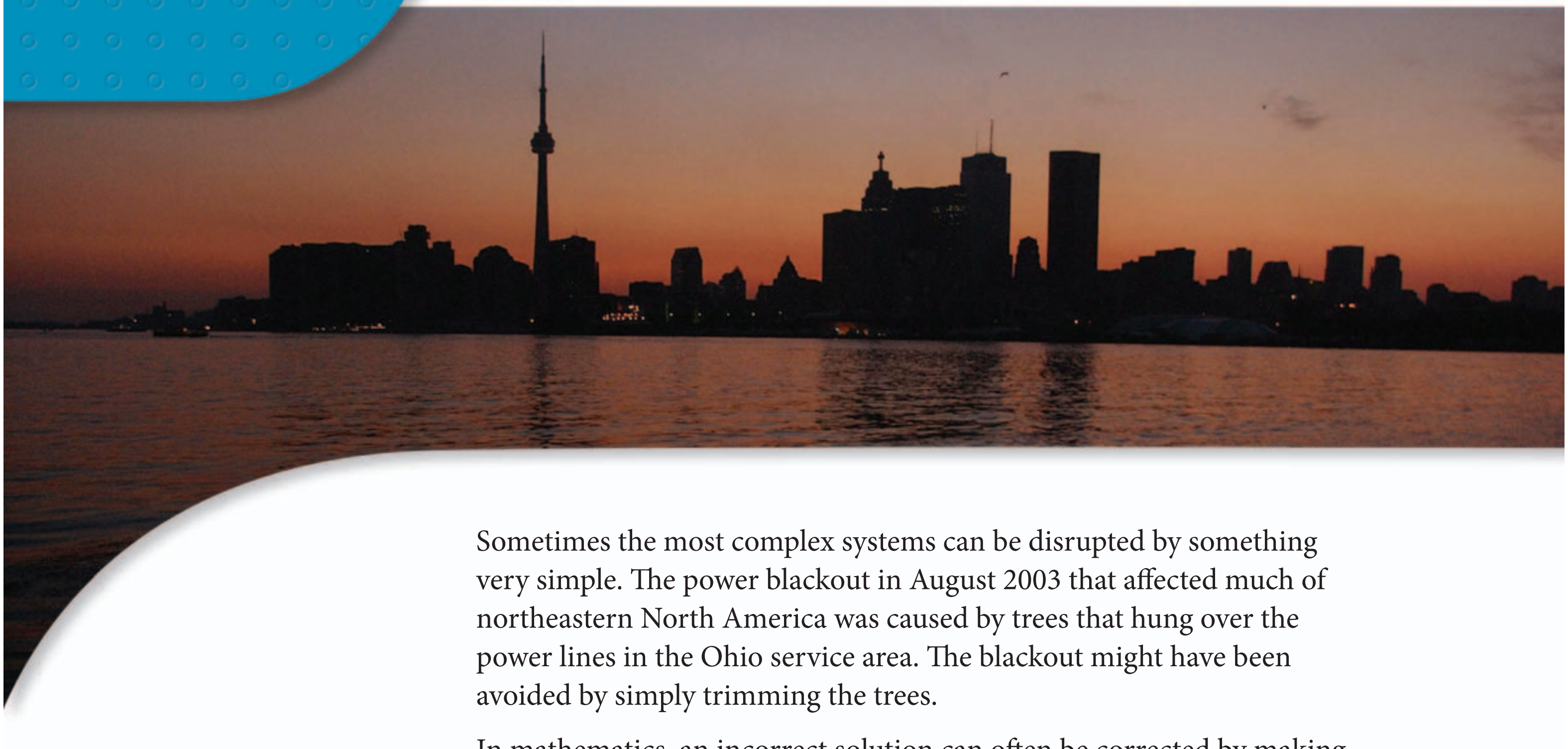
15. Some trinomials do not appear to be quadratic, but can be rewritten as quadratic expressions. For each trinomial, substitute $x^2 = s$ and $x^4 = s^2$, factor the trinomial that results, then substitute $s = x^2$.

a) $x^4 - 26x^2 + 25$

b) $x^4 - 53x^2 + 196$

c) $x^4 - 45x^2 + 324$

5.4 Factor Trinomials of the Form $ax^2 + bx + c$



Sometimes the most complex systems can be disrupted by something very simple. The power blackout in August 2003 that affected much of northeastern North America was caused by trees that hung over the power lines in the Ohio service area. The blackout might have been avoided by simply trimming the trees.

In mathematics, an incorrect solution can often be corrected by making a minor change, such as using the correct sign for a number or fixing a simple calculation error.

Investigate

Factor Trinomials of the Form $ax^2 + bx + c$

1. a) Copy and complete the table. Factor each trinomial by finding the greatest common factor. Then, write the trinomial factor as a product of its binomial factors.

Trinomial	Common Factored Form	Fully Factored Form
$3x^2 + 21x + 36$		
$2x^2 + 2x - 12$		
$5x^2 - 30x + 40$		
$-7x^2 - 21x - 14$		

b) Check your answers by expanding.

2. Refer to your answers to question 1. Use a similar method to factor each trinomial fully.

a) $4x^2 + 8x + 4$ **b)** $5x^2 - 5x - 30$

c) $-2x^2 - 6x + 20$ **d)** $-x^2 + 5x - 4$

3. Factor each trinomial.

a) $3x^2 - 3x + 18$ **b)** $2x^2 - 18x + 28$

4. Compare the trinomials and your answers from questions 2 and 3. Explain any similarities or differences.

5. Factor each trinomial, if possible.

a) $3x^2 + 13x - 10$ **b)** $12x^2 + 11x + 2$

6. Compare the trinomials and your answers from questions 2 and 5. Explain any similarities or differences.

7. **Reflect** Describe a method you could use to factor a trinomial of the form $ax^2 + bx + c$ where $a \neq 1$. Will this method always work? Use examples to explain.

Example 1

Factor $ax^2 + bx + c$

Factor the trinomial $4x^2 - 8x - 60$.

Solution

$4x^2 - 8x - 60$

$= 4[x^2 - 2x - 15]$ Divide each term by 4 to find the other factor.

$= 4(x + 3)(x - 5)$ Factor the simplified trinomial. Find two numbers whose product is −15 and whose sum is −2.

Example 2

Factor a Polynomial

Factor each polynomial.

a) $2x^2 - 50$ **b)** $-4.9t^2 + 19.6t$

Solution

a) $2x^2 - 50$

$= 2[x^2 - 25]$ Factor out the GCF, 2.

$= 2(x + 5)(x - 5)$ The second factor, $x^2 - 25$, is a difference of squares.

b) $-4.9t^2 + 19.6t$

$= -4.9t(t - 4)$ Factor out the GCF, $-4.9t$.

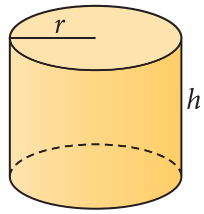
The expression $t - 4$ cannot be factored further.

In part b) of Example 2, it may not have been obvious that -4.9 was a common factor. When there is a coefficient on the x^2-term, always check to see if it is a common factor. Not all trinomials of the form $ax^2 + bx + c$ can be factored. For example, $2x^2 + 5x + 7$ becomes $2(x^2 + 2.5x + 3.5)$. The trinomial factor cannot be factored further since there is no pair of numbers whose sum is 2.5 and whose product is 3.5.

Example 3

Simplify Formulas by Factoring

The surface area of a cylinder is given by the formula S.A. $= 2\pi r^2 + 2\pi rh$.



a) Factor the expression for the surface area.

b) A cylinder has radius 3 cm and height 10 cm. Use both the original expression and the factored expression to find the surface area of this cylinder to the nearest square centimetre.

c) Describe a situation in which the factored form of the formula would be more useful than the original form.

Solution

a) Look for common factors in each term. Both terms have 2, π, and r as factors. So, $2\pi r$ is the greatest common factor.

$$\begin{aligned} \text{S.A.} &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h) \qquad \text{Factor out the GCF, } 2\pi r. \end{aligned}$$

b)
$$\begin{aligned} \text{S.A.} &= 2\pi r^2 + 2\pi rh \\ &= 2\pi(3)^2 + 2\pi(3)(10) \\ &\doteq 245.044 \end{aligned}$$

$$\begin{aligned} \text{S.A.} &= 2\pi r(r + h) \\ &= 2\pi(3)(3 + 10) \\ &= 6\pi(13) \\ &\doteq 245.044 \end{aligned}$$

The surface area of this cylinder is about 245 cm^2. Both methods gave the same result.

c) If you had to find the surface area for several cylinders, the factored form would be more useful. Fewer keystrokes are required for the factored form, so there is less chance of making a calculation error.

Math Connect

Your answer may be slightly different depending on the number of decimal places used for π. For example, if $\pi \doteq 3.14$ was used, the calculated surface area would be 244.92 cm^2.

Key Concepts

- To factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, first use common factoring to factor a out of each term. Then, express the trinomial factor as a product of binomial factors.
- When working with formulas, it is often useful to factor the expression to simplify calculations.

Discuss the Concepts

D1. Factor the trinomial $0.5x^2 + 4x + 6$ fully. How does the method you used to factor this trinomial, in which $a < 1$, compare to the method used in Example 1, in which $a > 1$? Explain.

D2. Describe a situation in which it is useful to factor an expression of a formula before using it to calculate measures.

Practise A

For help with questions 1 to 3, refer to Example 1.

1. Factor the common factor from each trinomial. Then, factor the trinomial factor. Expand to check.

a) $2x^2 + 16x + 30$ **b)** $4x^2 + 20x - 24$

c) $3x^2 + 18x + 15$ **d)** $2x^2 + 2x - 24$

e) $5x^2 + 5x - 10$ **f)** $3x^2 - 12x + 12$

2. Factor each trinomial fully. Expand to check.

a) $7x^2 - 77x + 210$ **b)** $6x^2 - 60x + 126$

c) $-3x^2 - 30x - 72$ **d)** $10x^2 - 140x - 320$

e) $-5x^2 + 50x - 105$ **f)** $-2x^2 + 4x + 96$

3. Factor each trinomial fully. Check your work.

a) $1.2x^2 - 8.4x - 36$ **b)** $-2.5x^2 - 30x - 80$

c) $3.4x^2 - 37.4x + 95.2$ **d)** $-4.6x^2 - 55.2x - 165.6$

For help with questions 4 to 6, refer to Example 2.

4. Factor each polynomial fully. Expand to check.

a) $5x^2 + 20x$ **b)** $3x^2 - 21x$ **c)** $-7x^2 + 49x$

d) $-15x^2 - 75x$ **e)** $8.2x^2 + 65.6x$ **f)** $-4.9x^2 + 44.1x$

5. Factor fully. Then, check your work.

a) $3x^2 - 27$ **b)** $6x^2 - 96$ **c)** $-3x^2 + 48$

d) $-8x^2 + 648$ **e)** $1.2x^2 - 30$ **f)** $-4.5x^2 + 162$

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6. Factor fully. Then, check your work.

a) $6x^2 + 48x + 96$
b) $5x^2 - 45$
c) $9x^2 - 27x$
d) $10x^2 - 50x - 240$
e) $-4x^2 + 196$
f) $-2x^2 + 18x$
g) $1.5x^2 + 4.5x - 27$
h) $-6.2x^2 + 396.8$

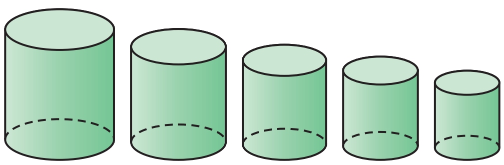
7. Which pairs contain equivalent expressions? How do you know?

a) $3x^2 + 30x + 75$ $\quad 3(x + 5)(x + 5)$
b) $5x^2 + 3x + 2$ $\quad 5(x + 2)(x + 1)$
c) $4x^2 - 10x + 24$ $\quad 4(x - 6)(x - 4)$
d) $-2x^2 - 22x - 40$ $\quad -2(x + 4)(x + 5)$

Apply B

For help with questions 8 and 9, refer to Example 3.

8. George is a designer for ZupperWare, a manufacturer of resealable tin containers with plastic lids. His latest design is a set of cylindrical containers that fit inside each other for easy storage.



The dimensions of the five containers are given in the table. The surface area of the tin portion of a container is given by the formula S.A. $= 2\pi rh + \pi r^2$.

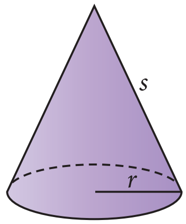
Height (cm)	Radius (cm)
20	10
18	9
16	8
14	7
12	6

a) Factor the expression for the surface area.

b) The exteriors of the containers are to be painted. Calculate the total surface area of the five containers.

c) The height of each container is double its radius. Explain how you could use this fact to further simplify the expression for the surface area from part a).

9. The surface area of a cone is given by the formula S.A. $= \pi r^2 + \pi rs$.

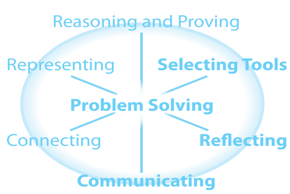


a) Factor the expression for the surface area.

b) Five cones all have a radius of 20 cm. Their slant height, s, is given in the table. Find the surface area of each cone.

Slant Height (cm)
40
45
50
55
60

c) A cone has a slant height that is three times its radius. Use your answer to part a) to write a simpler form of the expression for the surface area for this cone.



10. The makers of the Gateway Geyser in St. Louis, Missouri, claim that water is shot out of the fountain at 76 m/s and reaches heights of over 183 m. Ignoring air resistance, the height, h, in metres, of the water can be modelled by the relation $h = -4.9t^2 + 76t$, where t is the time, in seconds.

a) Factor the expression for the height of the water.

b) Make a table of values for times from 0 s to 10 s, in increments of 1 s. What is the approximate maximum height of the water (neglecting air resistance)?

c) Due to air resistance, the water only reaches about 65% of the predicted height. Is the manufacturer's claim regarding the maximum height of the fountain reasonable? Explain.

Chapter Problem

11. The fountain you are designing will use a pump that shoots water vertically out of the main fountain at 19.6 m/s and out of the smaller fountains at 14.7 m/s.

a) Write a relation for the height of the water, h, for each type of fountain using the general relation $h = -4.9t^2 + v_0t$, where v_0 is the initial velocity of the water and t is the time, in seconds.

b) Factor each expression from part a).

c) Find the approximate maximum height of the water from the main fountain and from the smaller fountains.

Achievement Check

12. Consider this pattern.

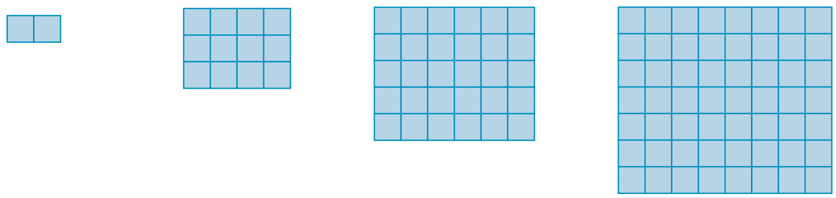


Diagram 1 **Diagram 2** **Diagram 3** **Diagram 4**

a) Count the number of unit squares in each diagram. Copy and complete the table.

Diagram	Number of Squares
1	
2	
3	
4	

b) The pattern continues. How many squares will be in Diagram 5?

c) Use first and second differences to determine if the relationship between the diagram number and the number of squares is quadratic.

d) Use a graphing calculator. Find an expression to represent the number of squares in a diagram.

e) Factor the expression from part d).

f) How do the factors of your expression relate to the side lengths of the rectangles?

Extend C

13. Expand and simplify each expression. Describe any patterns.

a) $(3x - 5)(3x + 5)$ **b)** $(4x + 7)(4x - 7)$ **c)** $(5x + 2)(5x - 2)$

14. Refer to your answer to question 13. Factor each expression.

i) $64x^2 - 9$ **ii)** $49x^2 - 36$ **iii)** $100x^2 - 9$

15. For a trinomial of the form $ax^2 + bx + c$ in which a is not a common factor, you can use the decomposition method to factor the trinomial. This method is shown below, using $6x^2 + 23x + 20$.

- Find two numbers whose product is ac and whose sum is b.

$$ac = 6 \times 20 \qquad b = 23$$
$$= 120$$
$$15 \times 8 = 120 \qquad 15 + 8 = 23$$

- Use the two numbers to "decompose" the middle term.

$$6x^2 + 23x + 20$$
$$= 6x^2 + 15x + 8x + 20$$

- Factor the first and second pairs of terms. The two binomial factors should be the same.

$$= 3x(2x + 5) + 4(2x + 5)$$

- Factor out the common binomial factor.

$$= (2x + 5)(3x + 4)$$

Factor each trinomial by decomposition. Expand to check.

a) $2x^2 + 19x + 24$

b) $10x^2 + 27x + 5$

c) $12x^2 + 13x + 3$

16. The quadratic relations in Group 1 have expressions that can be factored. Those in Group 2 have expressions that cannot be factored.

Group 1	Group 2
$y = x^2 + 5x + 4$	$y = x^2 + x + 1$
$y = 3x^2 - 27x + 54$	$y = -4x^2 - 10x - 8$
$y = -0.5x^2 + 3x + 8$	$y = 1.5x^2 - 5x + 8$

a) Graph each relation.

b) How are the graphs of the relations in Group 1 similar?

c) How are the graphs of the relations in Group 2 similar?

d) Compare the graphs of the relations in Group 1 to the graphs of the relations in Group 2. What do you notice? Explain.

5.5 The *x*-Intercepts of a Quadratic Relation

In the James Bond film, *The Man with the Golden Gun*, a complicated spiral car jump was filmed in only one take. The exact speed of the car had been determined by a computer, so the stunt could be performed precisely as planned. For stunts like this and others, it is important for a stunt coordinator to model not only where a car will begin to spiral and its general path, but also where it will land.

Investigate

Tools

- graphing calculator

Compare the Equation of a Quadratic Relation to Its Graph

1. a) Graph the quadratic relation $y = x^2 + 10x + 16$. What are the x-intercepts?
   b) Factor the expression on the right side of the relation.
   c) Compare the x-intercepts to the constant terms in the binomial factors of the factored form of the relation. What do you notice?
   d) Graph the factored relation in the same window. What do you notice?

2. Copy and complete the table. Find the x-intercepts without graphing.

Relation	Factored Relation	x-Intercepts
$y = x^2 + 10x + 21$		
$y = x^2 - 8x + 15$		
$y = x^2 + 2x - 24$		
$y = x^2 - 49$		

intercept form
- a quadratic relation of the form $y = a(x - r)(x - s)$
- the constants, r and s, represent the x-intercepts of the relation

zeros
- the x-coordinates of the points where the graph of a relation crosses the x-axis
- the x-intercepts of a relation
- the values of x for which $y = 0$

3. Graph each factored relation from question 2. Use the graph to find the x-intercepts. How do these x-intercepts compare to those you found in question 2?

4. **Reflect** Given a quadratic relation in the form $y = x^2 + bx + c$, how can you find the x-intercepts without graphing?

The relation $y = x^2 + 5x + 6$ can be expressed as $y = (x + 2)(x + 3)$. This is the **intercept form** of the quadratic relation. The x-intercepts are -2 and -3.

In general, to find the x-intercepts of a quadratic relation $y = ax^2 + bx + c$, first write the relation in intercept form, $y = a(x - r)(x - s)$. The x-intercepts are at $x = r$ and $x = s$. Since $y = 0$ at the points where the graph crosses the x-axis, the x-intercepts are also called the **zeros** of a quadratic relation.

Example 1

Factor to Find the Zeros of a Quadratic Relation

Factor each quadratic relation. Use the factors to find the zeros. Then, sketch the graph using the zeros and the y-intercept. Refer to the a-value to decide if the parabola opens upward or downward.

a) $y = 4x^2 + 4x - 168$

b) $y = -3x^2 + 24x - 48$

c) $y = x^2 - 8x$

d) $y = x^2 + 3x + 20$

Solution

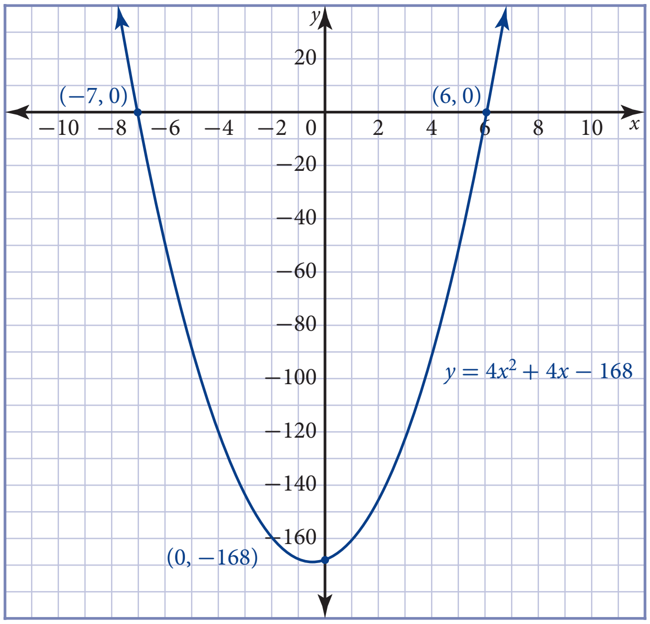
a) $y = 4x^2 + 4x - 168$

$= 4[x^2 + x - 42]$ Factor out the greatest common factor, 4.

$= 4(x + 7)(x - 6)$ Factor the resulting trinomial.

The zeros are at $x = -7$ and $x = 6$.

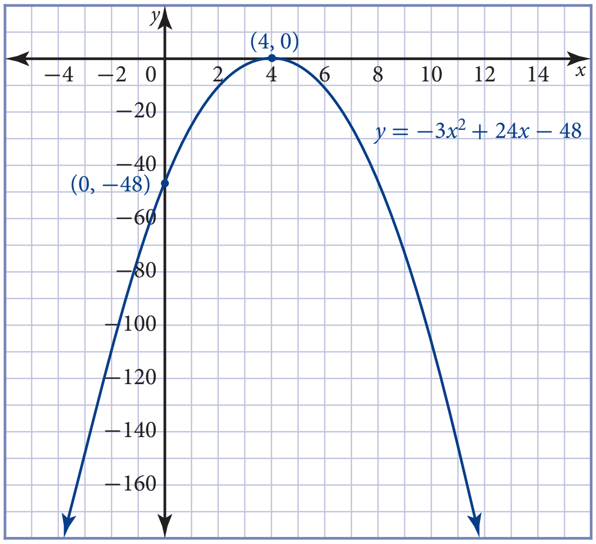
From the original relation, the y-intercept is -168. To sketch the graph of the relation, plot the zeros and the y-intercept. Since $a > 0$, the graph opens upward.



b) $y = -3x^2 + 24x - 48$
$= -3[x^2 - 8x + 16]$ Factor out the greatest common factor, -3.
$= -3(x - 4)(x - 4)$
$= -3(x - 4)^2$

Since both factors are the same, there is only one zero at $x = 4$.

The intercept form of this relation is the same as the vertex form. The vertex is (4, 0); this is also the x-intercept of the graph. From the original relation, the y-intercept is -48. Since $a < 0$, the graph opens downward.



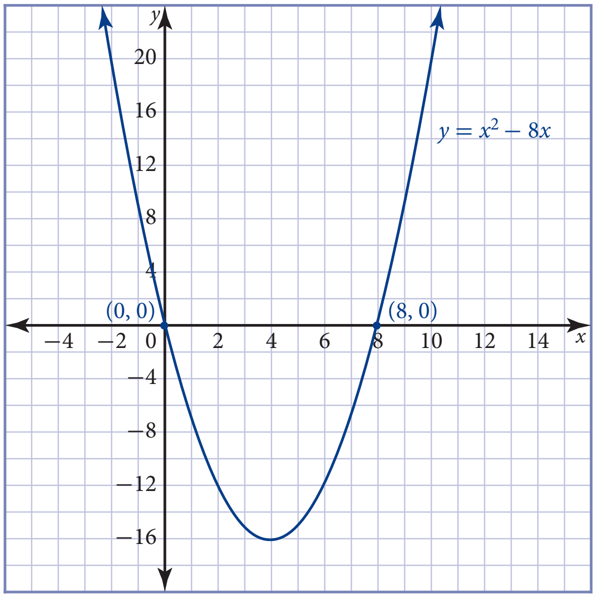
c) $y = x^2 - 8x$

$= x(x - 8)$ — The greatest common factor is x.

$= (x + 0)(x - 8)$ — You can also write the factors this way.

The zeros of this relation are at $x = 0$ and $x = 8$.

From the original relation, the y-intercept is 0. Since $a > 0$, the graph opens upward.

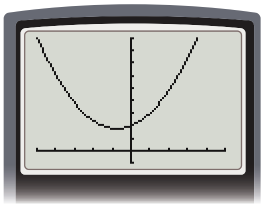


d) $y = x^2 + 3x + 20$

To factor this relation, try to find two values whose product is 20 and whose sum is 3.

There are no such values. There are two possible reasons for this:
- the relation has zeros that are not integers
- the relation has no zeros

Graph the relation to determine which reason applies to the relation.



The graph of $y = x^2 + 3x + 20$ does not cross the x-axis, so the relation has no zeros.

Example 2

Different Forms of a Quadratic Relation

Consider the quadratic relation $y = 3(x + 4)^2 - 108$.

a) What do you know about the graph of the given relation? Graph this relation.

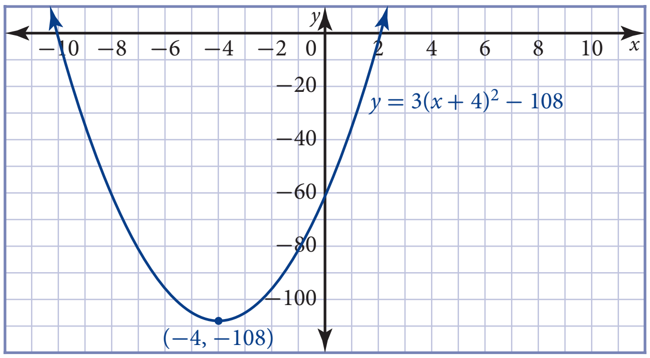
b) Write the relation in standard form. What do you know about the graph of the relation from the standard form? Graph the relation.

c) Write the relation in intercept form. What do you know about the graph of the relation given the intercept form? Graph the relation.

d) Compare the graphs of each form of the relation. Check your graphs using technology.

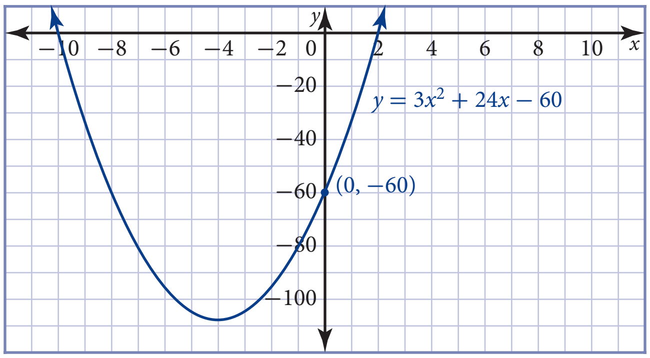
Solution

a) From the given relation, the vertex is at $(-4, -108)$. Since $a = 3$, the graph opens upward.



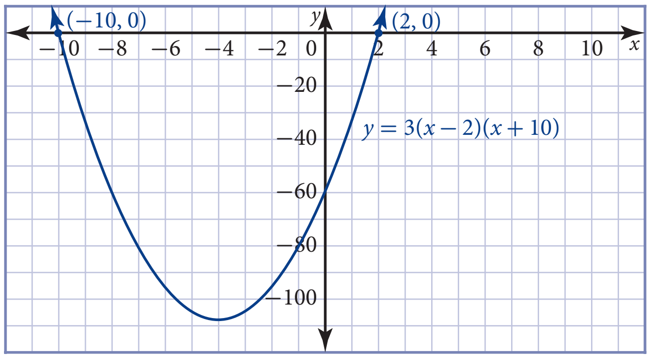
b)
$$\begin{aligned} y &= 3(x + 4)^2 - 108 \\ &= 3(x + 4)(x + 4) - 108 \\ &= 3[x^2 + 4x + 4x + 16] - 108 \\ &= 3x^2 + 12x + 12x + 48 - 108 \\ &= 3x^2 + 24x - 60 \end{aligned}$$

From the standard form, the y-intercept is -60.

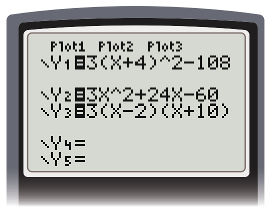


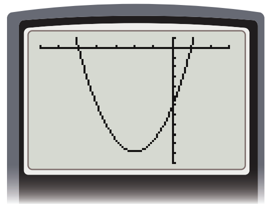
c) $y = 3x^2 + 24x - 60$
$= 3[x^2 + 8x - 20]$
$= 3(x - 2)(x + 10)$

From the intercept form, the zeros, or x-intercepts, are at $x = 2$ and $x = -10$.



d) The graphs are identical for all forms of the relation. Use a graphing calculator to check.





Example 3

Projectile Motion

A football is kicked from ground level. Its path is given by the relation $h = -4.9t^2 + 22.54t$, where h is the ball's height above the ground, in metres, and t is the time, in seconds.

a) Write the relation in intercept form.

b) Use the intercept form of the relation. Make a table of values with times from 0.5 s to 3.5 s, in increments of 0.5 s.

c) Use the intercept form of the relation to find the zeros.

d) Plot the zeros and the points from the table of values. Draw a smooth curve through the points.

e) When did the ball hit the ground? Explain how you found your answer.

Solution

a) There is no constant term. Factor out $-4.9t$.

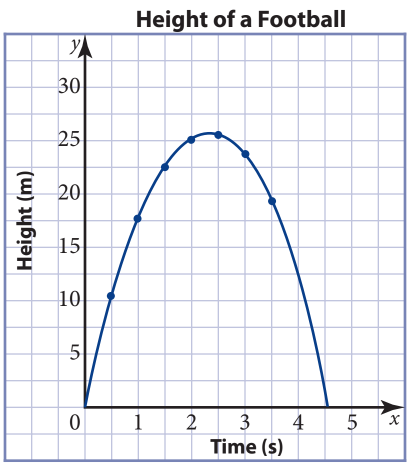
$$h = -4.9t^2 + 22.54t$$
$$= -4.9t(t - 4.6)$$

b)

Time (s)	Height (m)
0.5	10.045
1.0	17.640
1.5	22.785
2.0	25.480
2.5	25.725
3.0	23.520
3.5	18.865

c) The intercept form is $h = -4.9t(t - 4.6)$. The zeros are at $t = 0$ and $t = 4.6$.

d)



e) The zeros represent the times when the ball was on the ground. One of the zeros is at $t = 0$, which is when the ball was kicked. The other zero is at $t = 4.6$, which is when the ball landed.
The ball hit the ground at 4.6 s.

Key Concepts

- Given a quadratic relation in intercept form, $y = a(x - r)(x - s)$, the zeros, or x-intercepts, are r and s.
- The vertex, standard, and intercept forms of a quadratic relation give the same parabola when graphed.

Discuss the Concepts

D1. What do you know about the graph given each form of a quadratic relation?

a) vertex form, $y = a(x - h)^2 + k$

b) standard form, $y = ax^2 + bx + c$

c) intercept form, $y = a(x - r)(x - s)$

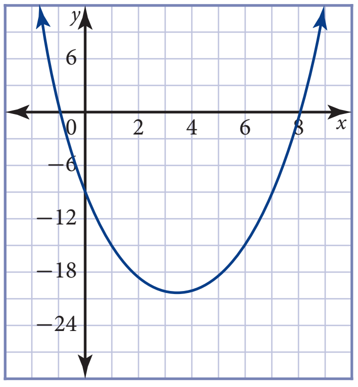
D2. The x-intercepts of a quadratic relation are at $x = -3$ and $x = 5$, and $a = 5$. Explain how you would find the standard form of the quadratic relation.

Practise A

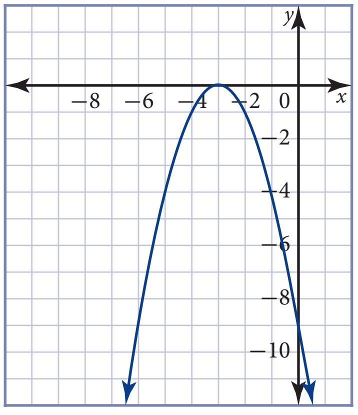
For help with questions 1 to 5, refer to Example 1.

1. Find the x-intercepts of each quadratic relation.

a)

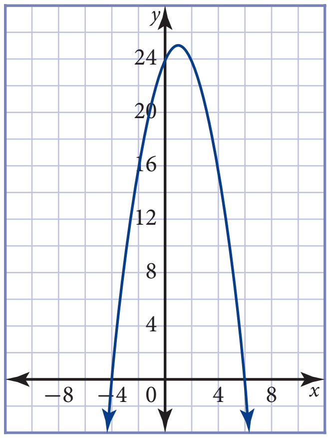


b)

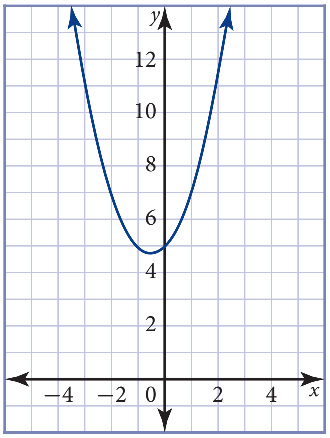


2. Find the zeros of each quadratic relation.

a)



b)



3. Find the zeros of each quadratic relation.

a) $y = (x - 5)(x + 3)$

b) $y = (x - 4)(x - 1)$

c) $y = 5(x - 9)(x - 9)$

d) $y = 3(x - 7)(x + 6)$

e) $y = -2(x + 8)(x + 2)$

f) $y = -3x(x + 5)$

4. Find the zeros by factoring. Check by graphing the intercept form and the standard form of each relation.

a) $y = x^2 + 10x + 16$

b) $y = x^2 - 2x - 35$

c) $y = x^2 - 6x - 7$

d) $y = 5x^2 - 125$

e) $y = 3x^2 + 39x + 108$

f) $y = 2x^2 - 28x + 98$

5. Find the zeros by factoring. Check by graphing the intercept form and the standard form of each relation.

a) $y = 4x^2 - 16x$

b) $y = 5x^2 - 125x$

c) $y = -5x^2 + 5x + 360$

d) $y = -x^2 - 18x - 81$

e) $y = -3.9x^2 + 19.5x$

f) $y = 7.5x^2 + 90x + 270$

For help with questions 6 and 7, refer to Example 2.

6. Which relations have more than one zero? Explain how you know.

a) $y = 3(x - 15)^2 + 2$

b) $y = -5(x + 2)^2 + 9$

c) $y = -(x - 8)^2 - 6$

d) $y = 9(x + 3)^2 - 10$

7. Given each quadratic relation in vertex form, express the relation in standard form and in intercept form. Then, check your answers by graphing all three forms.

a) $y = (x + 5)^2 - 4$

b) $y = (x - 3)^2 - 36$

c) $y = -2(x + 4)^2 + 8$

d) $y = 6(x + 2)^2 - 6$

e) $y = 3(x - 4)^2 - 48$

f) $y = -4(x - 5)^2 + 100$

Apply B

8. A skateboarder jumps a gap that is 1.3 m wide. Her path can be modelled by the relation $h = -1.25d^2 + 1.875d$, where h is her height above the ground and d is her horizontal distance from the edge of the gap, both in metres.

a) Write the relation in intercept form.

b) Determine the zeros of the relation. Will the skateboarder make it across the gap? Explain.

c) Copy and complete the table.

Horizontal Distance (m)	Height (m)
0	
0.25	
0.50	
0.75	
1.00	
1.25	
1.50	

d) Estimate the maximum height the skateboarder reached during her jump.

e) Graph the relation.

9. A second skateboarder jumps off a ledge. His path is modelled by the relation $h = -0.8d^2 + 0.8d + 1.6$, where h is his height above the ground and d is his horizontal distance from the ledge, both in metres.

a) What is the height of the ledge?

b) Factor to find the zeros of the relation.

c) At what point will the skateboarder land on the ground?

d) Graph the relation.

10. In a target game at an amusement park, players launch a beanbag toward a bucket using a mallet and a small seesaw. The path of a beanbag that lands directly in the bucket can be modelled by the relation $h = -0.65d^2 + 1.625d$, where h is the beanbag's height above the table and d is the beanbag's distance from the seesaw, both in metres.

a) Find the zeros of the relation.

b) How far is the bucket from the seesaw?

c) Find the beanbag's maximum height above the table to the nearest tenth of a metre.

11. The path of a stunt car can be modelled by the relation $h = -0.1d^2 + 0.5d + 3.6$ where h is the car's height above the ground and d is the car's horizontal distance from the edge of the ramp, both in metres.

a) Find the zeros of the relation.

b) How far from the ramp will the car land?

c) Suppose the stunt is done inside a sound studio with a ceiling height of 10 m. Will the car hit the ceiling? Explain your reasoning.

Extend C

12. A quadratic relation of the form $y = ax^2 + bx + c$ that cannot be factored might still have zeros. Another method for finding the zeros of a quadratic relation is to use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula to find the zeros of each relation.

a) $y = 3x^2 + 21x + 30$

b) $y = 16x^2 - 40x - 75$

c) $y = 2x^2 + 5x - 6$

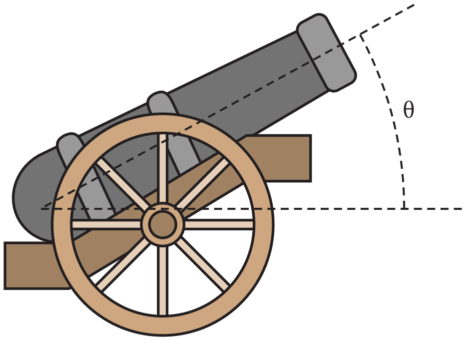
13. A quadratic relation of the form $y = ax^2 + bx + c$ has zeros if the expression $b^2 - 4ac$ in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is greater than or equal to zero. Determine if each relation has zeros.

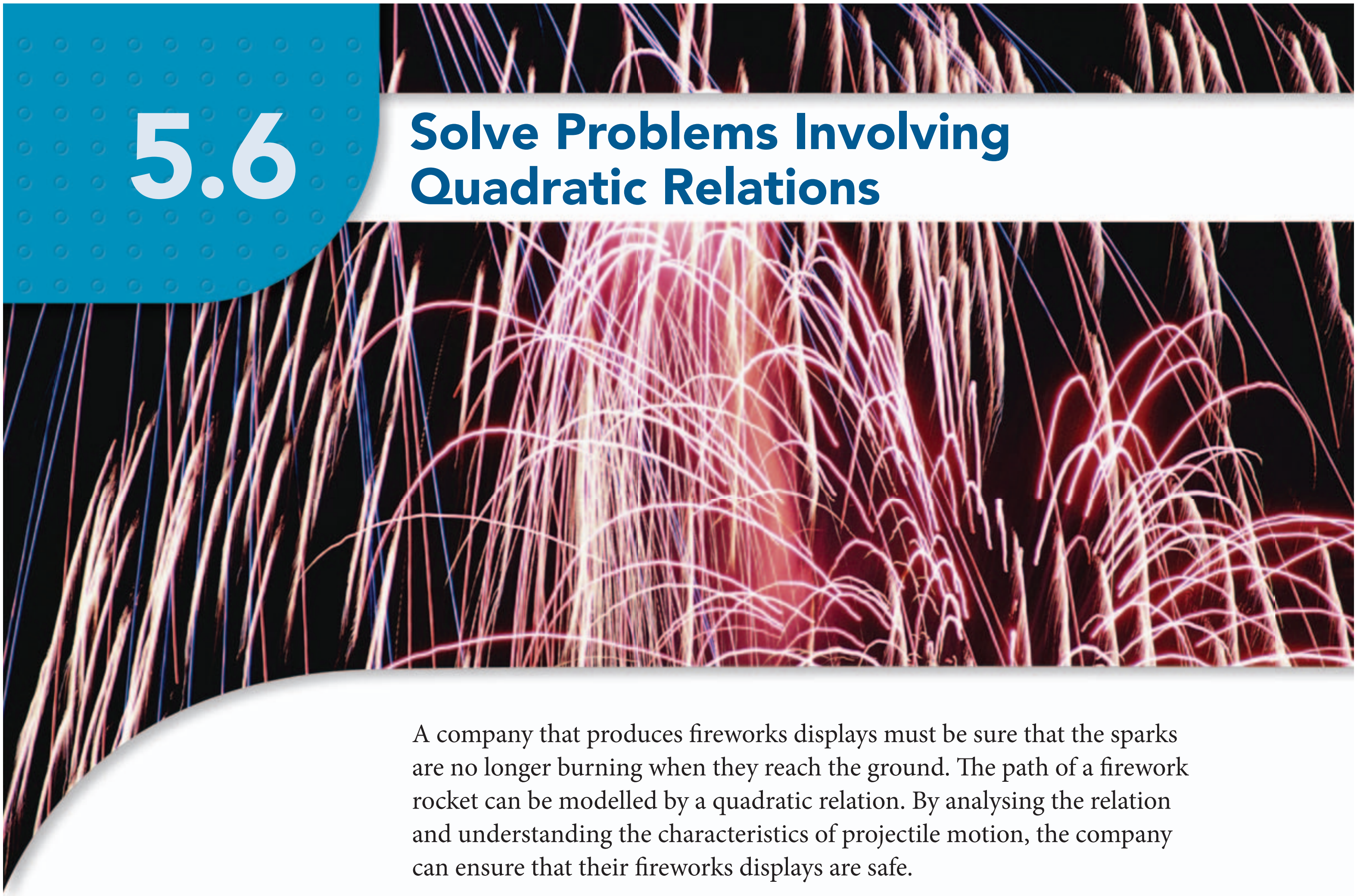
a) $y = 5x^2 + 3x + 15$

b) $y = 25x^2 + 60x + 36$

c) $y = 7x^2 - 10x + 5$

14. A cannonball is shot from ground level with an initial velocity of 20 m/s. Ignoring air resistance, its path can be modelled by the relation $h = -\frac{0.0125}{(\cos\theta)^2}d^2 + (\tan\theta)d$, where h is the cannonball's height above the ground, in metres, d is the horizontal distance from the cannon, in metres, and θ is the cannon's angle of elevation, in degrees. What angle of elevation will allow the cannonball to travel the greatest distance?





5.6 Solve Problems Involving Quadratic Relations

A company that produces fireworks displays must be sure that the sparks are no longer burning when they reach the ground. The path of a firework rocket can be modelled by a quadratic relation. By analysing the relation and understanding the characteristics of projectile motion, the company can ensure that their fireworks displays are safe.

Example 1

Use the Zeros to Find the Maximum

Consider the quadratic relation $y = 4x^2 - 72x + 260$.

a) Does this relation have a maximum or minimum? How do you know?

b) Find the zeros of the relation.

c) Determine the x-coordinate of the maximum or minimum point.

d) Find the maximum or minimum.

e) Write the relation in vertex form.

Solution

a) Since a is positive, the parabola opens upward. It has a minimum.

b) To find the zeros, factor the expression.

$$\begin{aligned} y &= 4x^2 - 72x + 260 \\ &= 4(x^2 - 18x + 65) \\ &= 4(x - 5)(x - 13) \end{aligned}$$

The zeros are at $x = 5$ and $x = 13$.

c) The minimum point occurs at the vertex. The x-coordinate of the vertex is midway between the zeros. It is the mean of the zeros.

$\frac{5 + 13}{2} = 9$ To find the mean of two numbers, add the numbers and divide the sum by 2.

The x-coordinate of the minimum point is 9.

d) Substitute $x = 9$.

$$\begin{aligned} y &= 4x^2 - 72x + 260 \\ &= 4(9)^2 - 72(9) + 260 \\ &= 324 - 648 + 260 \\ &= -64 \end{aligned}$$

The minimum is -64.

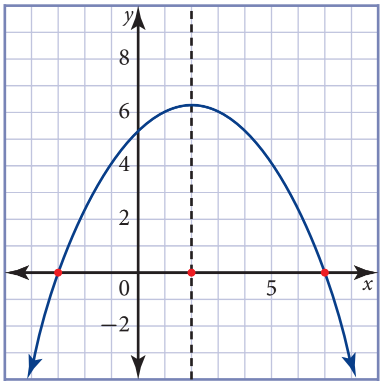
e) The minimum value, -64, occurs when $x = 9$. The coordinates of the vertex are $(9, -64)$. From the original relation, $a = 4$. Substitute the a-value and the coordinates of the vertex into $y = a(x - h)^2 + k$.

$y = 4(x - 9)^2 - 64$

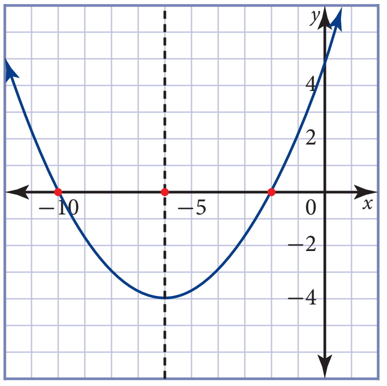
The relation in vertex form is $y = 4(x - 9)^2 - 64$.

axis of symmetry
- a vertical line through the vertex of a parabola
- the x-intercept of the vertical line is halfway between the zeros

The graph of a quadratic relation is symmetrical. The maximum or minimum lies halfway between the zeros, on the **axis of symmetry**, which is a vertical line through the vertex. The x-intercept of the axis of symmetry is midway between the zeros.



The zeros are at $x = -3$ and $x = 7$.
The equation for the axis of symmetry is $x = 2$.



The zeros are at $x = -10$ and $x = -2$.
The equation for the axis of symmetry is $x = -6$.

Example 2

Find the Width of a Walkway

A rectangular garden with dimensions 12 m by 8 m is surrounded by a walkway of uniform width, x. The total area of the garden and the walkway is 252 m^2.

a) Write expressions for the total length and the total width of the garden and walkway.

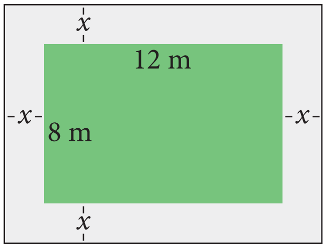
b) Write an expression for the total area of the garden and walkway.

c) Find the width of the walkway.

d) Check your answer to part c).

Solution

a) Draw and label a diagram to represent the situation.



$$\begin{aligned} \text{Length} &= x + 12 + x \\ &= 12 + 2x \end{aligned}$$

$$\begin{aligned} \text{Width} &= x + 8 + x \\ &= 8 + 2x \end{aligned}$$

The total length of the garden and the walkway is $12 + 2x$. The total width of the garden and the walkway is $8 + 2x$.

b)
$$\begin{aligned} \text{Area} &= \text{length} \times \text{width} \\ &= (12 + 2x)(8 + 2x) \\ &= 96 + 24x + 16x + 4x^2 \\ &= 4x^2 + 40x + 96 \end{aligned}$$

An expression for the total area of the garden and walkway is $4x^2 + 40x + 96$.

c) Substitute 252 for the area and find the zeros.

$$\begin{aligned} 252 &= 4x^2 + 40x + 96 \\ 252 - 252 &= 4x^2 + 40x + 96 - 252 \end{aligned}$$

To make the left side equal zero, subtract 252 from both sides.

$$\begin{aligned} 0 &= 4x^2 + 40x - 156 \\ 0 &= 4(x^2 + 10x - 39) \\ 0 &= 4(x - 3)(x + 13) \end{aligned}$$

From the factored form, the zeros are 3 and -13. Only the positive answer, 3, is realistic given the question. The walkway is 3 m wide.

d) You can check your answer to part c) two ways.

Substitute $x = 3$ into the expressions for length and width, then find the area.

$$\begin{aligned} \text{Length} &= 12 + 2x \\ &= 12 + 2(3) \\ &= 18 \end{aligned} \qquad \begin{aligned} \text{Width} &= 8 + 2x \\ &= 8 + 2(3) \\ &= 14 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \text{length} \times \text{width} \\ &= (18)(14) \\ &= 252 \end{aligned}$$

Substitute $x = 3$ into the expression for the total area.

$$\begin{aligned} \text{Area} &= 4x^2 + 40x + 96 \\ &= 4(3)^2 + 40(3) + 96 \\ &= 252 \end{aligned}$$

Example 3

Write the Equation of a Quadratic Relation

a) Graph each quadratic relation on the same set of axes.

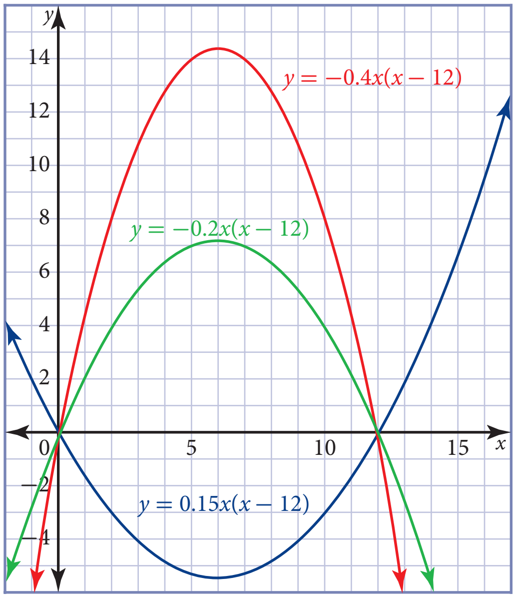
i) $y = 0.15x(x - 12)$ **ii)** $y = -0.2x(x - 12)$ **iii)** $y = -0.4x(x - 12)$

b) What is the same about the graphs and their relations? What is different?

c) Write the equation for a parabola that has the same zeros but passes through point (4, 8).

Solution

a)



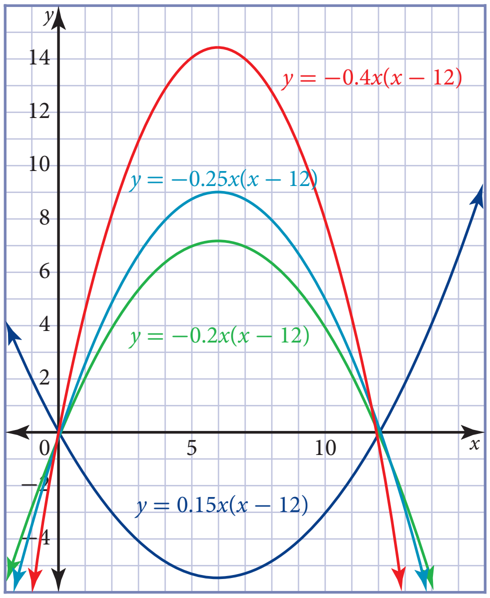
b) From the graphs, the parabolas have the same zeros: 0 and 12. Each parabola has a different vertical stretch or compression, and a different vertex. Two of the parabolas open downward and one opens upward. Each relation contains the expression $x(x - 12)$. The a-values of the relations are different.

c) Each relation has the form $y = ax(x - 12)$. Substitute the coordinates of the point that is to lie on the parabola.

Substitute $x = 4$ and $y = 8$.

$y = ax(x - 12)$
$8 = a(4)(4 - 12)$
$8 = a(4)(-8)$
$8 = -32a$
$a = -0.25$

The relation is $y = -0.25x(x - 12)$. The graph of this relation is the light blue parabola.



Key Concepts

- The maximum or minimum point of a parabola is halfway between the zeros of the parabola.
- The axis of symmetry is a vertical line through the vertex and a point on the x-axis halfway between the zeros.

Discuss the Concepts

D1. You have learned about three forms of quadratic relations:
- standard form, $y = ax^2 + bx + c$
- vertex form, $y = a(x - h)^2 + k$
- intercept form, $y = a(x - r)(x - s)$

What information about the parabola can you obtain from each form?

D2. Is it always possible to find the maximum or minimum of a quadratic relation using the zeros of the relation? Use examples to explain.

Practise A

1. Find the zeros of each quadratic relation.
   a) $y = (x + 4)(x - 5)$
   b) $y = (x + 9)(x + 15)$
   c) $y = 8(x + 3)(x + 19)$
   d) $y = 5(x - 8)(x + 10)$

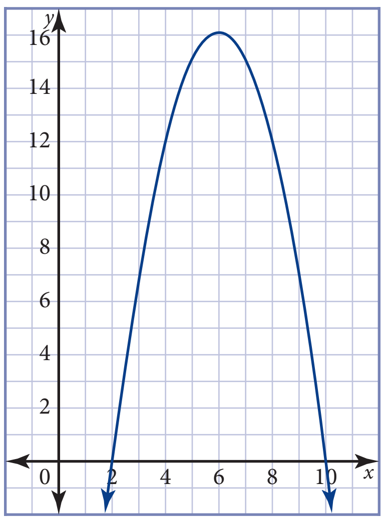
2. Express each quadratic relation in intercept form.
   a) $y = x^2 + 7x + 12$
   b) $y = x^2 + 11x + 28$
   c) $y = 3x^2 + 39x + 120$
   d) $y = -2x^2 + 10x + 132$

3. Find the zeros of each quadratic relation.
   a) $y = x^2 + 3x - 28$
   b) $y = x^2 - 16$
   c) $y = 2x^2 - 2x - 112$
   d) $y = 3x^2 + 21x - 294$
   e) $y = 5x^2 - 280$
   f) $y = -2x^2 + 18$
   g) $y = -4.9x^2 + 24.5x + 245$
   h) $y = 2.5x^2 + 50x - 560$

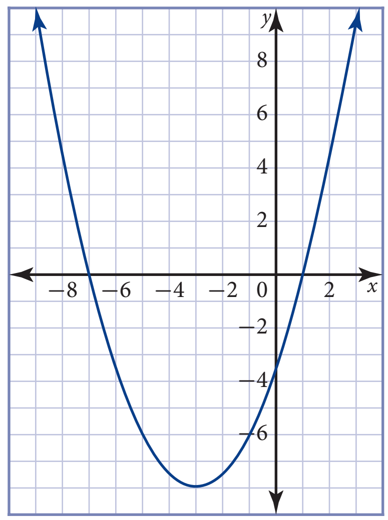
For help with questions 4 to 6, refer to Example 1.

4. What is the equation of the axis of symmetry for each parabola?

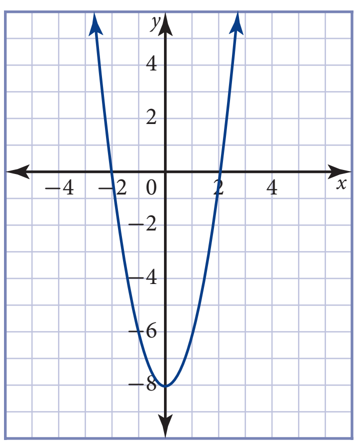
a)



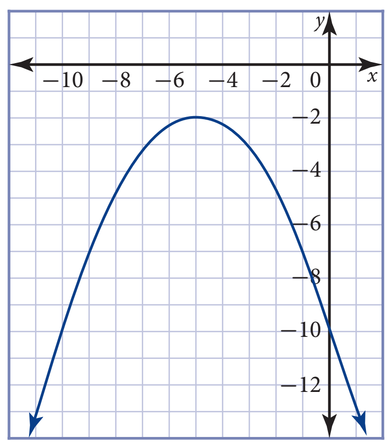
b)



c)



d)



5. Find the equation of the axis of symmetry for each quadratic relation.

a) $y = (x + 4)(x + 12)$

b) $y = (x - 7)(x - 1)$

c) $y = 8(x - 5)(x + 9)$

d) $y = -5(x + 12)(x - 4)$

e) $y = 6x(x + 10)$

f) $y = -3x(x - 8)$

6. Refer to question 5. Write each relation in standard form and in vertex form. Check your answers by graphing.

Apply B

For help with question 7, refer to Example 2.

7. A rectangular pool is 6 m wide and 10 m long. A concrete deck of uniform width is to surround the pool.

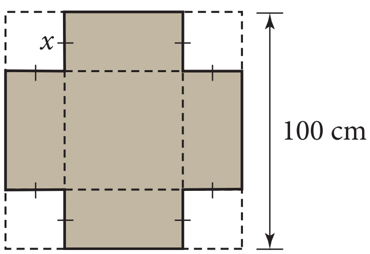
a) Sketch and label a diagram to represent the pool and the concrete deck.

b) Write an expression for the total width and the total length of the pool and deck.

c) Write a relation, in standard form, for the total area of the pool and deck.

d) If the total area of the pool and deck cannot exceed 320 m^2, what is the greatest possible width of the deck?

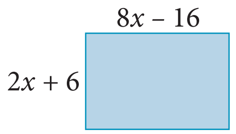
8. A square-based box with an open top is to be made from a square piece of cardboard that has side length 100 cm. The sides of the box are formed when four congruent square corner pieces are removed. The height of the box to be formed is represented by the value x.



a) Determine an expression for the area of cardboard used to make the box.

b) If the surface area of the box is to be 6400 cm^2, find the height of the box.

9. **a)** Write an expression for the area of this rectangle.



b) For what value of x will the rectangle have area 576 m^2?

For help with question 10, refer to Example 3.

10. **a)** Write three different quadratic relations, in standard form, for parabolas with zeros at $x = 1$ and $x = -5$.

b) Graph each relation.

c) Determine the quadratic relation for a parabola with the same zeros that passes through $(-3, -20)$.

11. A pattern of rectangles is made with unit squares. The relationship between the total number of unit squares, T, and the diagram number, d, is given by the relation $T = d^2 + d$.

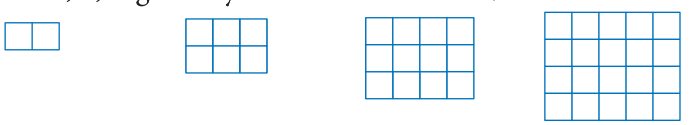
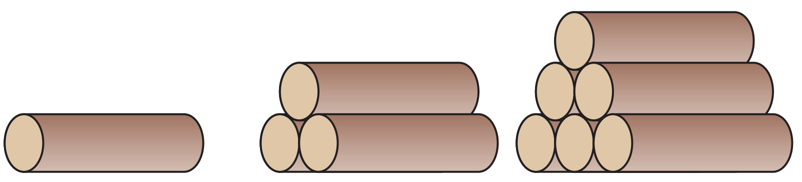


Diagram 1 **Diagram 2** **Diagram 3** **Diagram 4**

a) How many unit squares are needed for Diagram 8?

b) What diagram number would you expect to contain 110 unit squares?

12. Logs are stacked in a triangular pattern as shown. The relationship between the total number of logs, T, and the number of layers, L, is given by the relation $T = 0.5L^2 + 0.5L$.



1 layer **2 layers** **3 layers**

a) How many logs would be in 7 layers?

b) How many layers would there be for 120 logs?

c) Would a pile of 160 logs fit the triangular pattern? Explain.

13. A fireworks company is testing a new firework rocket. Once it explodes in the air, its path can be modelled by the relation $h = -4.9t^2 + 44.1t$, where h is the rocket's height, in metres, and t is the time, in seconds.

a) When will the rocket hit the ground?

b) What is the rocket's maximum height?

c) If the rocket continues to glow for 2.5 s after it begins to fall, will it be glowing when it hits the ground? Explain.

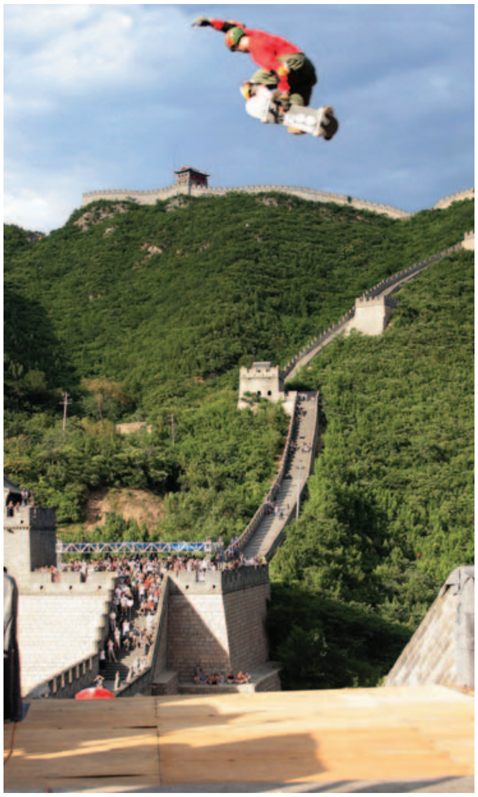
14. The speed of a turbine aircraft engine is controlled by a power setting, x. The length of time, t, in hours, that the engine will run on a given amount of fuel at power setting x is given by the relation $t = -0.2x^2 + 3.2x - 5.6$.

a) Find the zeros of the relation.

b) Find the coordinates of the vertex. What do the coordinates of the vertex represent in terms of this situation?

c) Check your answers to parts a) and b) by graphing the relation, using technology.

15. In July 2005, professional skateboarder Danny Way jumped over the Great Wall of China. His path can be modelled by the relation $h = -0.05d^2 + 1.15d$, where h is his height above the Great Wall and d is his horizontal distance from the take-off ramp, both in metres.



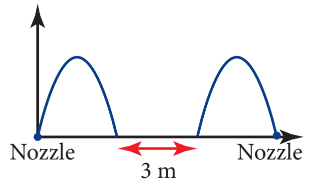
a) Factor the relation.

b) Use the factored relation to determine the distance between Danny's take-off and landing.

c) What was Danny's maximum height above the Great Wall?

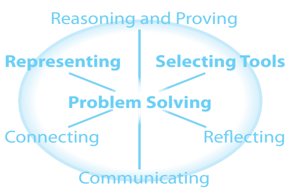
Chapter Problem

16. The fountain will have two identical jets of water side-by-side. The horizontal distance between the streams of water is 3 m. The path of the water from the jet on the left is modelled by the relation $h = -1.5(d - 1)^2 + 1.5$, where h is the height of the water and d is the horizontal distance from the nozzle, both in metres.



a) Find the horizontal distance from the left nozzle to where the water hits the ground.

b) Determine the horizontal distance between the nozzles for the two jets.



Achievement Check

17. A football player kicks a ball into the air. The ball's path can be modelled by the relation $h = -0.04(d - 19)^2 + 14.44$, where h is the ball's height and d is the ball's distance from the kicker, both in metres.

a) What is the ball's maximum height reached by the ball?

b) Express the relation in standard form and in intercept form.

c) What horizontal distance will the ball travel before it lands?

d) The goalposts are 35 m away and the crossbar is approximately 3 m high. Will the ball clear the crossbar?

Extend C

18. Ignoring air resistance, the path of a cannonball shot from ground level is modelled by the relation $h = -\frac{5}{(v_0 \cos \theta)^2}d^2 + (\tan \theta)d$, where h is the cannonball's height above the ground, in metres, d is the cannonball's horizontal distance from the cannon, in metres, v_0 is the cannonball's initial velocity, in metres per second, and θ is the cannon's angle of elevation, in degrees. What angle and initial velocity should be used to hit a target 90 m from the cannon?

5 Review

5.1 Expand Binomials, pages 234–241

1. Expand and simplify.

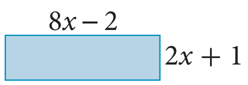
a) $(x + 5)(x + 8)$

b) $(2x + 9)(7x - 10)$

c) $(x + 13)^2$

d) $(x - 7)(x + 7)$

2. Write a simplified expression for the area of the rectangle.



5.2 Change Quadratic Relations From Vertex Form to Standard Form, pages 242–247

3. Write each relation in standard form.

a) $y = 5(x + 10)^2 + 7$

b) $y = -0.5(x + 8)^2 + 4$

c) $y = 9(x - 8)^2 - 4$

d) $y = 2(x + 1)^2 - 6$

4. Find the y-intercept for each relation in question 3.

5. A ball is kicked straight up. Its path is modelled by the relation $h = -4.9t^2 + v_0t + h_0$, where h is the ball's height in metres, h_0 is the ball's initial height, in metres, t is the time in seconds, and v_0 is the ball's initial velocity, in metres per second. The ball reaches a maximum height of 45 m after 3 s. Determine the ball's initial velocity and initial height.

5.3 Factor Trinomials of the Form $x^2 + bx + c$, pages 248–255

6. Factor.

a) $x^2 + 15x$

b) $x^2 + 13x + 40$

c) $x^2 + 10x + 25$

d) $x^2 - 81$

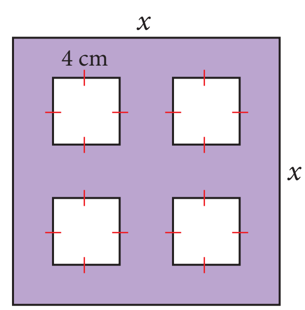
e) $x^2 + 2x - 24$

f) $x^2 - 12x + 35$

g) $x^2 - 100$

h) $x^2 - 11x - 12$

7. a) Write a factored expression for the area of the shaded region of this figure.



b) Calculate the area of the shaded region when $x = 30$ cm.

5.4 Factor Trinomials of the Form $ax^2 + bx + c$, pages 256–263

8. Factor fully.

a) $4x^2 + 72x + 308$

b) $12x^2 + 96x$

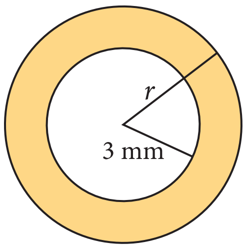
c) $3x^2 - 12x - 135$

d) $-2x^2 - 24x - 72$

e) $-8x^2 + 200$

f) $10x^2 - 80x - 200$

9. a) Write a factored expression for the area of the shaded region of this figure.



b) Suppose $r = 15$ mm. Find the area of the shaded region.

5.5 The x-Intercepts of a Quadratic Relation, pages 264–275

10. Find the zeros of each quadratic relation.

a) $y = x^2 - 16x$

b) $y = x^2 - 16$

c) $y = 6x^2 + 24x - 192$

11. Write each quadratic relation in standard form, then find the zeros.

a) $y = 3(x - 1)^2 - 147$

b) $y = -4(x + 6)^2 + 36$

12. The path of a soccer ball can be modelled by the relation $h = -0.1d^2 + 0.5d + 0.6$, where h is the ball's height and d is the horizontal distance from the kicker.

a) Find the zeros of the relation.

b) What do the zeros mean in this context?

5.6 Solve Problems Involving Quadratic Relations, pages 276–285

13. For each quadratic relation, find the zeros and the maximum or minimum.

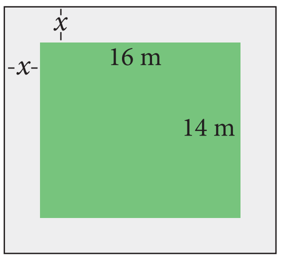
a) $y = x^2 + 16x + 39$

b) $y = 5x^2 - 50x - 120$

c) $y = -2x^2 - 28x + 64$

d) $y = 6x^2 + 36x - 42$

14. A garden is to be surrounded by a paved border of uniform width.



a) Write a simplified expression for the area of the border.

b) The border is to have an area of $216\ m^2$. Find the width of the border.

15. A rider on a mountain bike jumps off a ledge. Her path is modelled by the relation $h = -0.3d^2 + 1.2d + 1.5$, where h is her height above the ground and d is her horizontal distance from the ledge, both in metres.

a) What is the height of the ledge?

b) How far was the rider from the ledge when she landed?

5 Practice Test

For questions 1 to 6, choose the best answer.

1. Which expression is equivalent to $(2x + 9)(2x + 9)$?

A $4x^2 + 81$

B $4x^2 - 81$

C $4x^2 + 18x + 81$

D $4x^2 + 36x + 81$

2. Which expression is the result of expanding and simplifying $(5x - 7)(3x + 5)$?

A $15x^2 + 46x + 35$

B $15x^2 + 4x - 35$

C $8x^2 + 20x + 13$

D $8x^2 - 13$

3. Which relation represents the same parabola as $y = 5(x - 6)^2 - 20$?

A $y = 5x^2 - 6x - 20$

B $y = 5x^2 - 12x + 16$

C $y = 5x^2 - 60x + 160$

D $y = 5x^2 - 12x + 160$

4. Which expression is the factored form of $x^2 - 8x - 20$?

A $(x - 8)(x - 20)$

B $(x - 10)(x + 2)$

C $(x + 8)(x + 20)$

D $(x - 2)(x + 10)$

5. Which is the equation of the axis of symmetry for the quadratic relation $y = (x - 7)(x + 17)$?

A $x = -5$ **B** $x = 7$

C $x = 12$ **D** $x = 17$

6. Which are the zeros for the quadratic relation $y = 5x^2 - 1125$?

A $x = 0$

B $x = 5, x = 15$

C $x = -15, x = 5$

D $x = -15, x = 15$

7. Which expression is the factored form of $4x^2 - 44x - 240$?

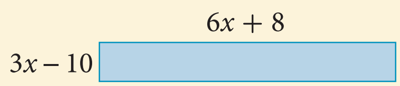
A $4(x - 44)(x - 240)$

B $4(x - 4)(x - 60)$

C $4(x - 15)(x + 4)$

D $4(x - 11)(x - 60)$

8. **a)** Write an expression, in simplified form, for the area of the rectangle.



b) Find the area of the rectangle when $x = 5$ cm.

9. Write each quadratic relation in standard form.

a) $y = 13(x + 7)^2 + 11$

b) $y = -4(x - 3)^2 + 16$

c) $y = 5.6(x - 1.2)^2 - 8.2$

10. Find the zeros of each quadratic relation.

a) $y = x^2 - 2x - 35$

b) $y = 3x^2 + 12x - 96$

c) $y = -2.5x^2 - 40x - 70$

Chapter Problem Wrap-Up

Throughout this chapter, you looked at many aspects of designing and building a fountain. Now, you will design your own fountain. Describe how you would use quadratic relations in the design. Besides the jets of water, what other aspects of the fountain must you consider?

11. The curve of a cable on a suspension bridge can be modelled by the relation $h = 0.0025(d - 100)^2 + 25$ where h is the cable's height above the ground and d is the horizontal distance from the tower, both in metres.

a) At what height does the cable meet the tower?

b) What is the least height of the cable above the ground?

12. A circus acrobat jumps off a raised platform. He lands on a trampoline at stage level below. His path can be modelled by the relation $h = -0.7d^2 + 0.7d + 4.2$, where h is his height above the stage and d is his horizontal distance from the edge of a platform, both in metres.

a) What is the height of the platform?

b) How far from the edge of the platform did the acrobat land?

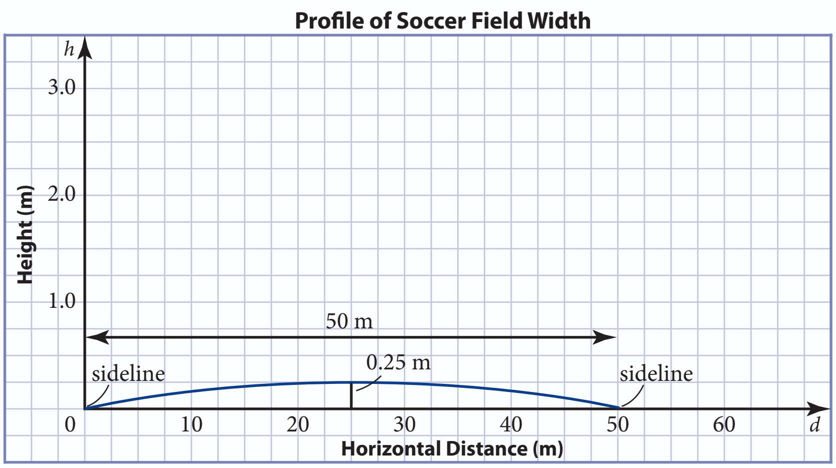
c) What was the acrobat's maximum height above the stage?

Task

Design a Soccer Field

Have you ever walked on a new football or soccer field? If so, you probably noticed that the field was slightly arched, not flat. These fields are highest in the centre, permitting rainwater to drain away quickly.

1. The graph shows the profile of the width of a soccer field, viewed from one end. Assume that the cross-section is parabolic. Write a quadratic relation that models the profile of the soccer field. Let h represent the height, in metres, above the sidelines, and d represent the horizontal distance, in metres, from the left sideline.



2. When viewed lengthwise, the cross-section of the soccer field has the same parabolic shape and the same 0.25 m rise in the middle of the field, but the goal lines are 100 m apart. Write a quadratic relation that models this profile of the soccer field.

3. The spot where penalty kicks are taken is 11 m in front of the goal line. How high above the goal line is the penalty spot, to the nearest tenth of a centimetre?

4. A sprinkler system is to be installed in the field. Precise positioning of the piping and sprinkler heads is necessary so the heads are flush with the ground. How far from the sidelines should a sprinkler head along the halfway line be placed so that it is flush with the ground 0.16 m above the side?

5. The centre circle in the middle of the field has a radius of 9.15 m. What are the minimum and maximum heights above the sideline along the centre circle, to the nearest tenth of a centimetre?