# 4 Quadratic Relations I

Gaming software designers use mathematics and physics to make figures and vehicles move realistically. Often, the motion can be modelled by a quadratic relation.

In this chapter you will extend your knowledge of quadratic relations by connecting their equations and graphs in real world settings.

## In this chapter, you will


- construct tables of values and graph quadratic relations arising from real-world applications
- determine and interpret meaningful values of the variables, given a graph of a quadratic relation arising from a real-world application
- determine, through investigation using technology, and describe the roles of $a$, $h$, and $k$ in quadratic relations of the form $y = a(x - h)^2 + k$ in terms of transformations on the graph of $y = x^2$
- sketch graphs of quadratic relations represented by the equation $y = a(x - h)^2 + k$

## Key Terms

| | |
|---|---|
| mathematical model | parabola |
| maximum | vertex |
| minimum | vertex form |

Min is an industrial furniture designer. A four-year course at the Ontario College of Art and Design gave her the skills to create products that perform well, look good, and meet the economic needs of her clients. She uses her skills in mathematics to design and produce her furniture and also to calculate her designs' marketability and cost.

# Prerequisite Skills

## Number Skills

**1.** Add.

**a)** $3.4 + 9.7$ **b)** $1.3 + (-3)$

**c)** $-11.3 + 3.6$ **d)** $-4.8 + (-12.3)$

**2.** Subtract.

**a)** $8.8 - 15.3$ **b)** $17.5 - (-8.6)$

**c)** $-4.5 - 6.0$ **d)** $-10 - (-3.3)$

**3.** Multiply.

**a)** $(4.5)(9.2)$ **b)** $(6.3)(-4)$

**c)** $(-10)(13)$ **d)** $(-7.1)(-1.5)$

## Algebraic Expressions

**4.** Simplify.

**a)** $3x + (-5x)$

**b)** $9x^2 - (-10) + 3x - 2x^2$

**c)** $3x^2 - 4x + 2x + 5x^2$

**d)** $-5x^2 + 2x - (3x^2 - 2)$

**5.** Find the value of $y$ when $x = 0$.

**a)** $y = 3x^2$

**b)** $y = -9x^2 + 6$

**c)** $y = 6(x + 4)^2$

**d)** $y = -3(x + 8)^2 - 10$

**6.** Find the value of $y$ when $x = 3$.

**a)** $y = -2x^2$ **b)** $y = 5x^2 + 2$

**c)** $y = -4(x - 8)^2$ **d)** $y = (x + 9)^2 + 7$

**7.** Find the value of $y$ when $x = -4$.

**a)** $y = 12x^2$

**b)** $y = -11x^2 - 7$

**c)** $y = -3(x + 15)^2$

**d)** $y = 9(x - 12)^2 - 20$

## Linear Relations

**8.** Copy and complete each table of values.

**a)** $y = 3x - 4$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**b)** $y = -0.5x + 21$

| x | y |
|---|---|
| 8 | |
| 12 | |
| 16 | |
| 20 | |
| 24 | |

**9.** Refer to question 8.

**a)** Graph each relation.

**b)** Identify the slope and the $y$-intercept for each graph.

**10.** Copy and complete each table. Find the first differences for each set of data.

**a)**

| x | y | First Differences |
|---|---|---|
| 3 | 8 | |
| 4 | 15 | |
| 5 | 23 | |
| 6 | 31 | |
| 7 | 39 | |

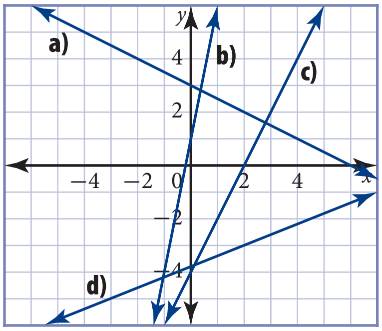
**b)**

| x | y | First Differences |
|---|---|---|
| 13 | 0 | |
| 14 | 1 | |
| 15 | 4 | |
| 16 | 9 | |
| 17 | 16 | |

**c)**

| x | y | First Differences |
|---|---|---|
| −5 | 3 | |
| −4 | 3 | |
| −3 | 3 | |
| −2 | 3 | |
| −1 | 3 | |

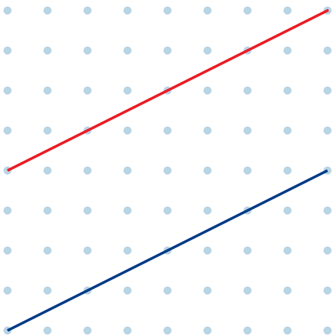
**11.** Identify the slope and the $y$-intercept of each line.



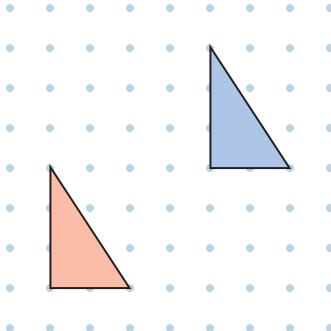

## Transformations

**12.** Describe a transformation that would move each red figure onto its image.

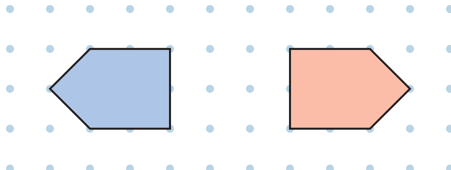
**a)**



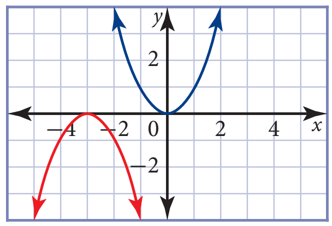
**b)**



**c)**

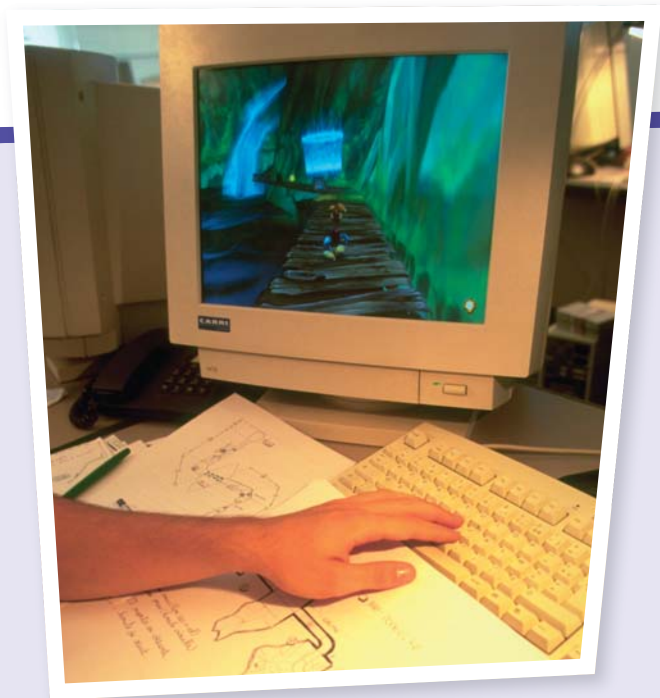


**d)**




## Chapter Problem

You have just been hired by Gamerz Inc. to develop a new extreme sports simulation game. Which extreme sports would you include in your game? What types of motion will you need to model?



# 4.1 Modelling With Quadratic Relations

Quadratic relations can be used to represent the shape of a suspension bridge or the path of water from a fountain. The ancient Babylonians (3000 B.C.E.) studied quadratic relations in the context of farming. Quadratic relations were the subject of a debate in the House of Commons in London, England, in 2003. Why do you think the British Parliament would discuss quadratic relations?

## Investigate

### Tools

- calculator

**Math Connect**

1 ha = 10 000 m$^2$

### Develop a Mathematical Model

Several farmers have square fields of different sizes. They want to know how much fertilizer to buy given that six bags of fertilizer cover one hectare (ha).

**1.** Copy and complete the table.

| Side Length of Square Farm (m) | Area of Farm (ha) | Bags of Fertilizer Needed |
|---|---|---|
| 100 | 1 | 6 |
| 200 | | |
| 300 | | |
| 400 | | |
| 500 | | |
| 600 | | |

Remember, the dependent variable is plotted on the vertical axis. The amount of fertilizer needed depends on the size of the field.

**2.** Draw a graph comparing the bags of fertilizer needed to the area of the fields. Describe the shape of the graph.

**3.** **a)** Draw a graph comparing the bags of fertilizer needed to the side length of the fields. Describe the shape of the graph.

**b)** How many bags of fertilizer are needed for a square field with a side length of 1200 m?

**c)** Describe how you could find the number of bags of fertilizer needed for a field with side length $n$.

**4.** **Reflect** Refer to the table and your graphs from questions 1 to 3. How can you use the table to determine if the relation is linear or non-linear?

**mathematical model**
- a mathematical description of a real situation
- can be a diagram, a graph, a table of values, a relation, a formula, a physical model, or a computer model

**parabola**
- a symmetrical U-shaped curve
- the graph of a quadratic relation

In the Investigate, you used a **mathematical model** to represent the relationship between the side length of the field and the number of bags of fertilizer needed. On a graph, this model is represented by half of a **parabola**. Since negative side lengths for fields do not make sense, the graph does not show the other half of the parabola, on the left side of the vertical axis.

## Example 1

### Use a Graph to Identify a Quadratic Relation

The table shows a soccer ball's height above the ground over time after it was kicked in the air.

| Time (s) | Height (m) |
|---|---|
| 0 | 0.10 |
| 0.5 | 7.80 |
| 1.0 | 12.00 |
| 1.5 | 13.80 |
| 2.0 | 13.00 |
| 2.5 | 9.75 |
| 3.0 | 4.00 |

**a)** Graph the data. Draw a smooth curve through the points.

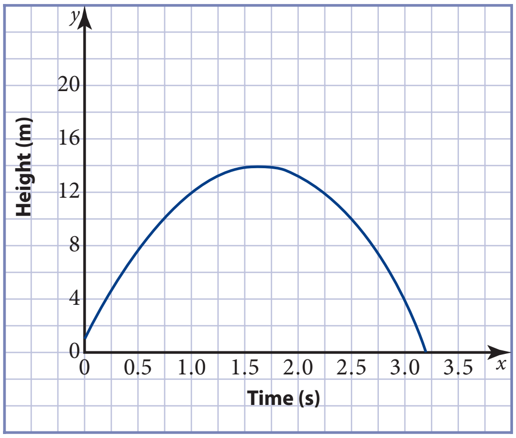
**b)** Describe the shape of the graph.

**c)** What was the ball's maximum height?

**d)** For about how many seconds was the ball in the air?

## Solution

**a)** Time is the independent variable, so plot it on the horizontal axis.




**b)** The graph is a parabola that opens downward.

**c)** The ball reached a maximum height of about 13.80 m.

**d)** The ball was in the air for about 3 s.

**vertex**
- the lowest point on a parabola that opens upward, or the highest point on a parabola that opens downward

**minimum**
- on an *x*–*y* plane, the *y*-coordinate of the lowest point on a parabola that opens upward
- the *y*-coordinate of the vertex

**maximum**
- on an *x*–*y* plane, the *y*-coordinate of the highest point on a parabola that opens downward
- the *y*-coordinate of the vertex

The **vertex** of a parabola is the highest point if the parabola opens downward. The vertex is the lowest point if the parabola opens upward. In general, it is the point at which the graph changes from decreasing to increasing or from increasing to decreasing. A parabola always has a **minimum** or a **maximum**.

## Example 2

## Use Patterns to Identify a Quadratic Relation

The table shows how two variables, $x$ and $y$, are related.

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 6 |
| 2 | 9 |
| 3 | 10 |
| 4 | 9 |
| 5 | 6 |
| 6 | 1 |

**a)** Calculate the first and second differences.

**b)** Is the relation linear or quadratic? Explain.

## Solution

**a)**

| x | y | First Differences | Second Differences |
|---|---|---|---|
| 0 | 1 | | |
| | | $6 - 1 = 5$ | |
| 1 | 6 | | $3 - 5 = -2$ |
| | | $9 - 6 = 3$ | |
| 2 | 9 | | $1 - 3 = -2$ |
| | | $10 - 9 = 1$ | |
| 3 | 10 | | $-1 - 1 = -2$ |
| | | $9 - 10 = -1$ | |
| 4 | 9 | | $-3 - (-1) = -2$ |
| | | $6 - 9 = -3$ | |
| 5 | 6 | | $-5 - (-3) = -2$ |
| | | $1 - 6 = -5$ | |
| 6 | 1 | | |

Remember, for any quadratic relation, second differences are constant.

**b)** First differences are not constant, so the relation is not linear. Second differences are constant, so the relation is quadratic.

## Example 3


## Use Algebra to Identify a Quadratic Relation

**a)** Graph each relation.

**i)** $y = 3x + 4$

**ii)** $y = 2x^2 - 5x + 3$

**iii)** $y = -4x^2 + 3x - 4$

**iv)** $y = -0.5x^2 - 2$

**b)** Which of the relations are quadratic?

**c)** Look at the graphs that are parabolas. What do the equations of these relations have in common?

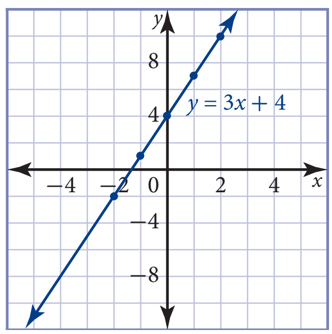
## Solution

**a) Method 1: Use Pencil and Paper**

- Create a table of values for each relation. Use the same $x$-values for each relation.
- Plot the points. Draw a line or a smooth curve through the points.

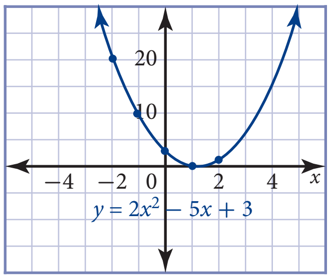
**i)** $y = 3x + 4$

| x | y |
|---|---|
| −2 | −2 |
| −1 | 1 |
| 0 | 4 |
| 1 | 7 |
| 2 | 10 |



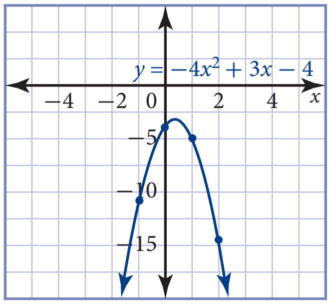
**ii)** $y = 2x^2 - 5x + 3$

| $x$ | $y$ |
|---|---|
| $-2$ | 21 |
| $-1$ | 10 |
| 0 | 3 |
| 1 | 0 |
| 2 | 1 |



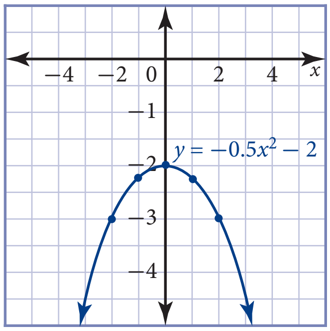

**iii)** $y = -4x^2 + 3x - 4$

| $x$ | $y$ |
|---|---|
| $-2$ | $-26$ |
| $-1$ | $-11$ |
| 0 | $-4$ |
| 1 | $-5$ |
| 2 | $-14$ |




**iv)** $y = -0.5x^2 - 2$

| $x$ | $y$ |
|---|---|
| $-2$ | $-4$ |
| $-1$ | $-2.5$ |
| 0 | $-2$ |
| 1 | $-2.5$ |
| 2 | $-4$ |




**Technology Tip**

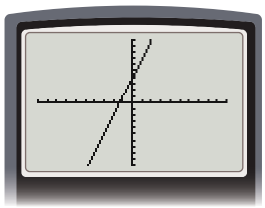
^ means "raised to the exponent"

**Method 2: Use a Graphing Calculator**

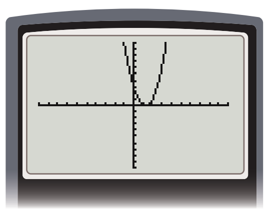
Clear all equations from the [Y=] window and turn off any stat plots.

- Press 2nd [STATPLOT] **4:PlotsOff** ENTER.
- Press Y=. Enter the first equation in Y1. Press ZOOM **6:Standard** to graph the first relation. For the other relations, just press GRAPH.
- Sketch each graph in your notes and label it with its relation.

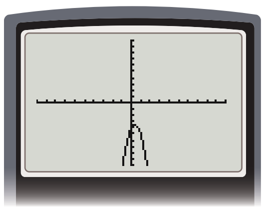
**i)**



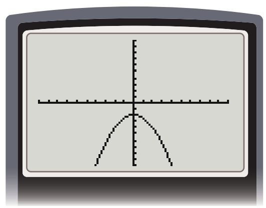
**ii)**



**iii)**



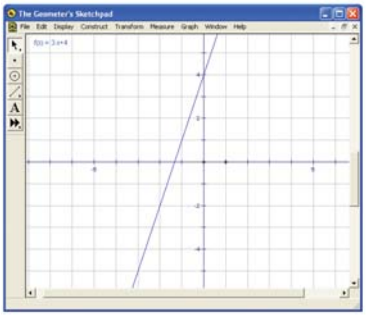
**iv)**



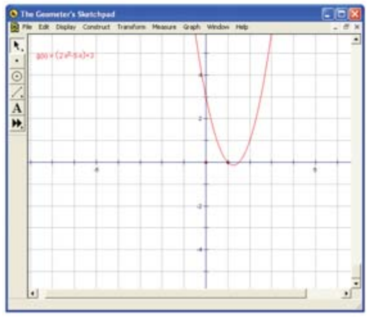
**Method 3: Use *The Geometer's Sketchpad®***

- Open a **New Sketch**. From the **Graph** menu, choose **Plot New Function**.
- In the **New Function** window, type $3x + 4$. Select **OK**.
- From the **Display** menu, choose **Color**. Select a unique colour for that graph and relation. Deselect the relation and the graph.
- Repeat these steps to graph the other relations.

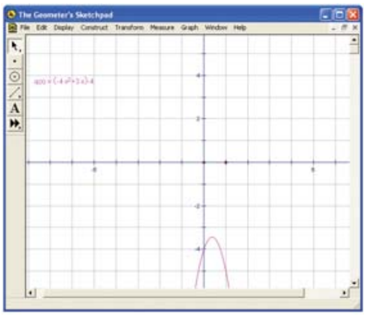
**i)**



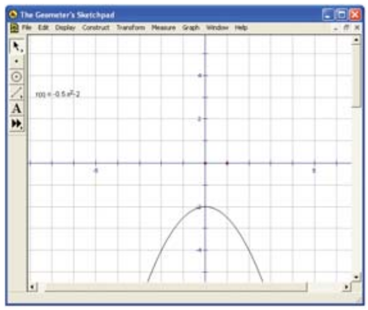

**ii)**




**iii)**




**iv)**




**b)** $y = 2x^2 - 5x + 3$, $y = -4x^2 + 3x - 4$, and $y = -0.5x^2 - 2$ result in parabolas when graphed. These are quadratic relations.

**c)** Each equation that results in a parabola has an $x^2$-term.

## Key Concepts

- The graph of a quadratic relation is a parabola.
- For any quadratic relation, second differences are constant.
- Every quadratic relation has an $x^2$-term; the degree of the polynomial is 2.

### Discuss the Concepts

**D1.** How can you use the table of values for a relation to determine if the relation is linear or quadratic?

**D2.** Write two different quadratic relations. Explain why they are quadratic.

## Practise A

**For help with question 1, refer to Example 1.**

1. Graph each relation. Determine if the relation is linear, quadratic, or neither.

a)

| x | y |
|---|---|
| −3 | −32 |
| −1 | −12 |
| 1 | 0 |
| 3 | 4 |
| 5 | 0 |
| 7 | −12 |
| 9 | −32 |

b)

| x | y |
|---|---|
| 0 | −3 |
| 1 | 5 |
| 2 | 13 |
| 3 | 21 |
| 4 | 29 |
| 5 | 37 |
| 6 | 45 |

c)

| x | y |
|---|---|
| 5 | −18 |
| 6 | −11 |
| 7 | −10 |
| 8 | −9 |
| 9 | −2 |
| 10 | 17 |
| 11 | 54 |

d)

| x | y |
|---|---|
| 2 | 73 |
| 4 | 97 |
| 6 | 97 |
| 8 | 73 |
| 10 | 25 |
| 12 | −47 |
| 14 | −143 |

e)

| x | y |
|---|---|
| −2 | 0 |
| −1 | −15 |
| 0 | −16 |
| 1 | −9 |
| 2 | 0 |
| 3 | 5 |
| 4 | 0 |

f)

| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |
| 25 | 5 |
| 36 | 6 |

**For help with question 2, refer to Example 2.**

2. Which of these relations are quadratic? How do you know?

a)

| x | y |
|---|---|
| −30 | 250 |
| −29 | 241 |
| −28 | 232 |
| −27 | 223 |
| −26 | 214 |
| −25 | 205 |
| −24 | 196 |

b)

| x | y |
|---|---|
| 18 | 0 |
| 20 | 3 |
| 22 | 4 |
| 24 | 4 |
| 26 | 0 |
| 28 | −5 |
| 30 | −12 |

c)

| x | y |
|---|---|
| 3 | 128 |
| 6 | 200 |
| 9 | 288 |
| 12 | 392 |
| 15 | 512 |
| 18 | 648 |
| 21 | 800 |

d)

| x | y |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |

**For help with question 3, refer to Example 3.**

**3.** **a)** Predict which relations are quadratic. Explain your reasoning.

**i)** $y = 14x^2 - 5x + 7$ **ii)** $y = -8x + 5$

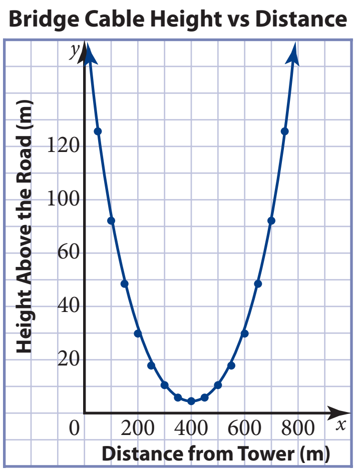
**iii)** $y = 3x^2 + 2$ **iv)** $y = 2^x$

**v)** $y = 3 + 2x - 15x^2$ **vi)** $y = 4 + x$

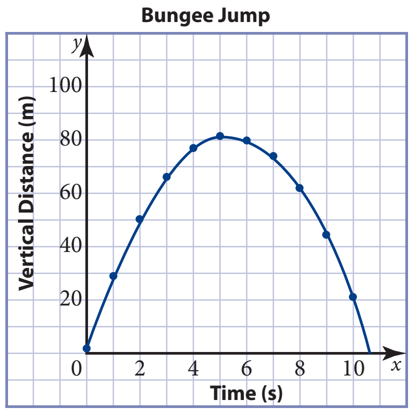
**b)** Check your predictions by graphing each relation.

**4.** Does each graph have a maximum or minimum value? Use the graph to estimate the maximum or minimum.

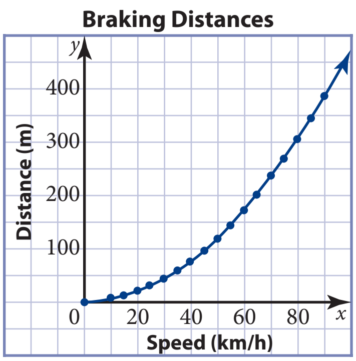
**a)**




**b)**




**c)**




## Apply B

**5.** Bonita and Carl ran a race. They used a CBR™ to measure their distance over time.

| Time (s) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Bonita's Distance (m) | 0.00 | 2.50 | 5.00 | 7.50 | 10.00 | 12.50 |
| Carl's Distance (m) | 0.00 | 0.75 | 3.00 | 6.75 | 12.00 | 18.75 |

**a)** Graph the data for Bonita and Carl on the same set of axes.

**b)** Which runner's distance–time relationship is quadratic? Explain.

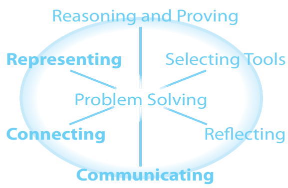
**6.** A cannonball is shot horizontally from the top of a cliff. Its path can be modelled by the relation $h = 150 - 4.9t^2$, where $h$ is the cannonball's height above the ground, in metres, and $t$ is the time, in seconds.

**a)** Copy and complete the table.

**b)** Is the relation quadratic? How do you know?

**c)** Graph the relation.

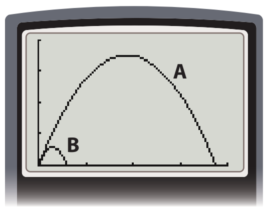
| Time (s) | Height (m) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |




**7.** Two balls are thrown upward at 15 m/s: one on Earth, near sea level, and one on the moon. The path of the ball on Earth is given by the relation $h = -4.9t^2 + 15t$, where $h$ is the ball's height above the ground, in metres, and $t$ is the time, in seconds. The path of the ball on the moon is given by the relation $h = -0.8t^2 + 15t$.

**a)** Create a table of values for each relation.

**b)** Refer to the graphing calculator screen shown below. Which curve models each relation? Justify your answer.




**Literacy Connect**

**8.** These words contain the prefix "quad."
- Quadrilateral: a polygon with four sides
- Quadriceps: a muscle divided into four parts that unite in a single tendon at the knee
- Quadruped: a four-legged animal

**a)** What does the prefix "quad" mean?

**b)** Explain how the word *quadratic* relates to the relation $y = ax^2 + bx + c$.

9. A farmer wants to use 100 m of fencing to build a small rectangular pen for his llamas. He would like the pen to have the greatest possible area.

   a) Copy and complete the table. Provide six possible sets of dimensions for the pen.

| Length (m) | Width (m) | Perimeter (m) |
|---|---|---|
| 40 | 10 | $2(40) + 2(10) = 100$ |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |

   b) Add a fourth column to the table. Calculate the area of each pen.

   c) Draw a graph to compare length and area.

   d) Use the graph to determine the dimensions of the pen with the greatest possible area.

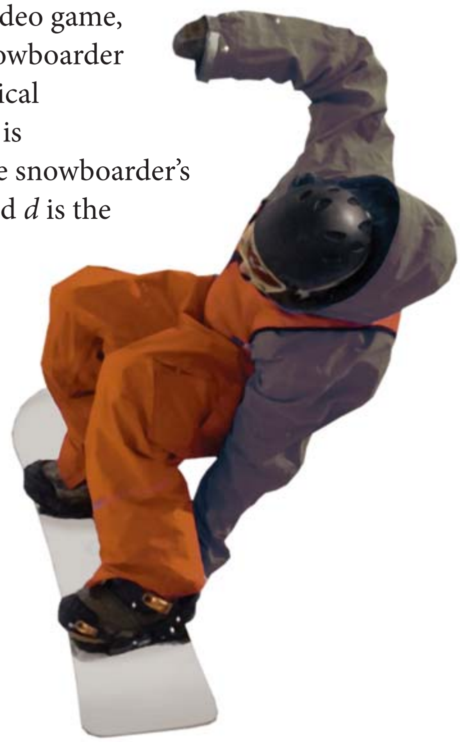
**Chapter Problem**

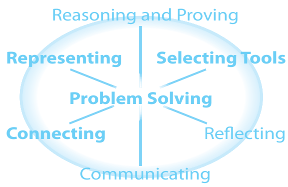
10. For part of a new extreme sports video game, you have to model the path of a snowboarder jumping off a ledge. The mathematical model developed from a video clip is $h = -0.05d^2 + 11.25$ where $h$ is the snowboarder's height above the base of the cliff and $d$ is the snowboarder's horizontal distance from the base of the cliff, both in metres.

   a) Create a table of values for the relation. Choose consecutive $d$-values.

   b) Graph the relation.

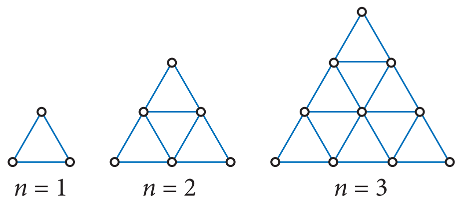
   c) At what horizontal distance from the cliff will the snowboarder land?






## Achievement Check

**11.** Toothpicks can be arranged to create equilateral triangles, where $n$ is the number of toothpicks on one side.




**a)** Draw triangles that have 4, 5, and 6 toothpicks on one side.

**b)** Copy and complete the table. Graph the relation between side length and total number of toothpicks.

| Side length, $n$ | Total number of toothpicks, $T$ | First Differences | Second Differences |
|---|---|---|---|
| 0 | 0 | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |

**c)** Use first and second differences to determine if the relationship between the side length of a triangle and the total number of toothpicks is quadratic. Explain how your graph supports your answer.

**d)** How many toothpicks are needed to build a triangle with a side length of 20 toothpicks?

**e)** What is the side length of the largest triangle that can be made with 200 toothpicks?

## Extend C

12. A police officer is parked on the side of the road. She sees a speeding car. The distance between the speeding car and the spot where the police car was parked is given by the relation $d = 20t$, where $d$ is the distance, in metres, and $t$ is the time, in seconds. The officer accelerates to catch the speeding car. Her distance from the spot where she was parked is given by the relation $d = 1.5t^2$. When will the officer catch up with the speeder? How far will she be from the spot where she was parked?

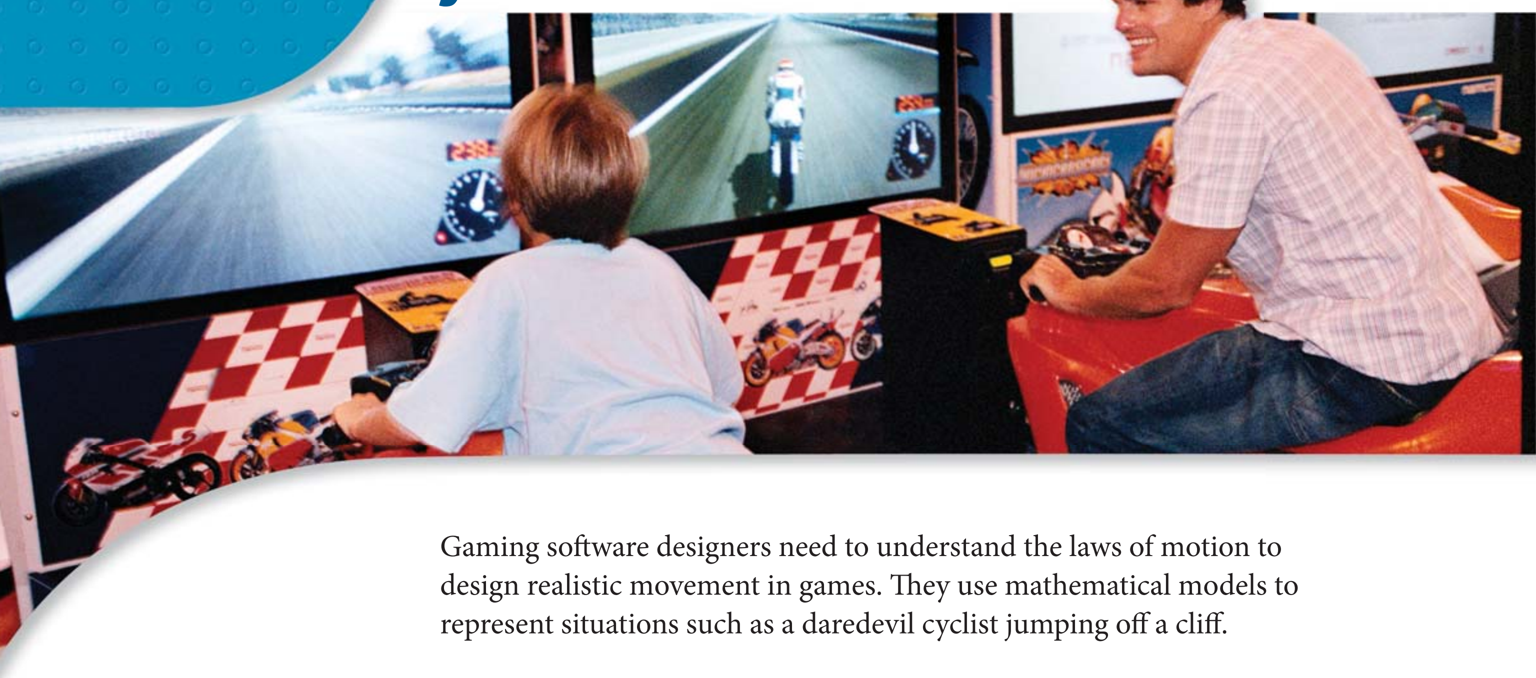
13. Suzy challenges Oliver to a 100-m race. Suzy runs and Oliver rides his bicycle. Suzy's speed is modelled by the relation $d = 3t$ and Oliver's speed is modelled by the relation $d = 0.1t^2$. For both, $d$ is the distance, in metres, and $t$ is the time, in seconds. Use a graph to determine who will win the race.

14. a) For the relation $y = x^3$, make a table of values and graph the relation. Is this relation quadratic? Explain why or why not.
    b) For the relation $y = x^4$, make a table of values and graph the relation. Is the relation quadratic? Explain why or why not.

**Literacy Connect**

15. Many people think that the St. Louis Gateway Arch in St. Louis, Missouri, is in the shape of a parabola. However, it is actually in the shape of a catenary [ka-*tee*-na-ree]. Research the characteristics of a catenary and find other examples of this shape.

# 4.2 The Quadratic Relation $y = ax^2 + k$



Gaming software designers need to understand the laws of motion to design realistic movement in games. They use mathematical models to represent situations such as a daredevil cyclist jumping off a cliff.

## Investigate

### Tools

- graphing calculator

## Graphs of $y = ax^2 + k$

**Method 1: Use a Graphing Calculator**

Clear any equations in the [Y=] window and turn off any stat plots.

**1.** Start by graphing $y = x^2$.
- Press [Y=] and enter $x^2$ for Y1.
- Use the left arrow key and move the cursor as far left as it will go. Change the graph's appearance by repeatedly pressing [ENTER] until a dotted line appears.
- Press [ZOOM] **6:ZStandard** to graph the relation in the standard window.
- As you add new relations, keep the graph of $y = x^2$ for comparison.

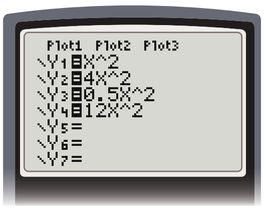
**Part A: The Effect of Changing *a***

**2. a)** Graph each relation.

**i)** $y = 4x^2$

**ii)** $y = 0.5x^2$

**iii)** $y = 12x^2$




**b)** How are the graphs the same? How are they different?

**c)** Press [2nd] [TABLE]. Compare the $y$-values for corresponding $x$-values. What do you notice?

3. **a)** Graph each relation.
   **i)** $y = -3x^2$
   **ii)** $y = -0.2x^2$
   **iii)** $y = -10x^2$

   **b)** How are the graphs the same? How are they different?

4. Write an equation for a parabola that
   **a)** opens downward
   **b)** opens upward

5. **a)** Graph each relation.
   **i)** $y = 0.1x^2$
   **ii)** $y = 0.05x^2$
   **iii)** $y = 0.6x^2$

   **b)** How are the graphs the same? How are they different?

6. **Reflect** Graph the relation $y = ax^2$ for $a$-values between 0 and 1. What effect does changing $a$ have on the graph as $a$ gets closer to zero?

7. **a)** Graph each relation.
   **i)** $y = 2x^2$
   **ii)** $y = 5x^2$
   **iii)** $y = 8.2x^2$

   **b)** How are the graphs the same? How are they different?

8. **Reflect** Graph the relation $y = ax^2$ for $a$-values greater than 1. What effect does changing $a$ have on the graph as $a$ increases?

9. Compare the graphs of $y = x^2$ and $y = ax^2$. For what values of $a$ is the graph of $y = ax^2$
   **a)** narrower than the graph of $y = x^2$?
   **b)** wider than the graph of $y = x^2$?

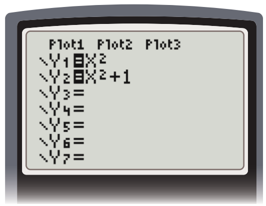
10. When the shape of a parabola is described, it is often compared to the graph of $y = x^2$, using terms such as "vertically stretched" (narrower) or "vertically compressed" (wider). Write an equation for a parabola that is
    **a)** vertically stretched and opens upward
    **b)** vertically stretched and opens downward
    **c)** vertically compressed and opens upward
    **d)** vertically compressed and opens downward

### Part B: The Effect of Changing *k*

Clear all relations except $y = x^2$.

**11. a)** Graph the relation $y = x^2 + 1$.

**b)** How does the graph of $y = x^2 + 1$ compare to the graph of $y = x^2$?




**12.** The relation $y = x^2 + 1$ is in the form of $y = x^2 + k$. What happens to the graph if the value of $k$ increases? Graph the relation using different positive values of $k$.

**13. Reflect** Graph the relation using different negative values of $k$. How does the graph change as the value of $k$ decreases?

**14.** Describe the position of the graph of $y = x^2 + k$ relative to the $x$-axis when the value of $k$ is

**a)** positive

**b)** negative

**c)** zero

> **Math Connect**
>
> A translation is one type of transformation. It is a slide. Other transformations include reflections and rotations.

**15.** One way to describe the position of the graph of $y = x^2 + k$ is to say that its vertex is translated upward or translated downward relative to the $x$-axis. Write an equation for a parabola with a vertex

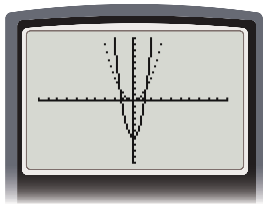
**a)** translated above the $x$-axis

**b)** translated below the $x$-axis

**c)** at the origin

### Part C: The Effects of Changing *a* and *k*

The effects of changing $a$ and $k$ can be seen in a new equation $y = ax^2 + k$.



**16.** Write an equation for each parabola. Graph each relation.

**a)** vertically stretched, opens upward, vertex is 4 units below the $x$-axis

**b)** vertically compressed, opens downward, vertex is 2 units above the $x$-axis

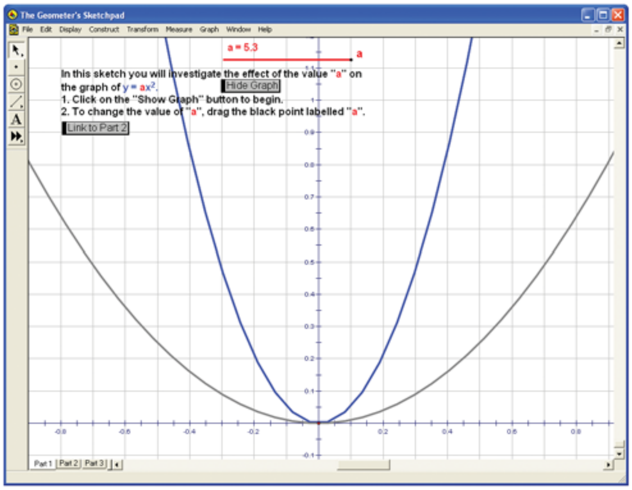
**c)** vertically stretched, opens downward, vertex is at the origin

**17. Reflect** You are given a graph of a parabola and must write its equation in the form $y = ax^2 + k$. Which value is easier to determine, $a$ or $k$? Explain.

## Tools

- computer
- *The Geometer's Sketchpad®*
- 4.2 Investigation.gsp

### Method 2: Use *The Geometer's Sketchpad®*




Go to *www.mcgrawhill.ca/links/foundations11* and follow the links to 4.2. Download the file **4.2 Investigation.gsp**. Open the sketch.

### Part A: The Effect of Changing *a*

**1.** Select **Show Graph** and drag point *a*. How does changing the value of *a* affect the shape of the blue parabola?

**2.** **Reflect** Compare the blue parabola to the graph of $y = x^2$ when $a$ is

**a)** positive **b)** negative

**3.** Write an equation for a parabola that

**a)** opens downward **b)** opens upward

**4.** **Reflect** Compare the blue parabola to the graph of $y = x^2$ when $a$ is

**a)** greater than 0 but less than 1

**b)** greater than 1

**c)** less than 0 but greater than $-1$

**d)** less than $-1$

**5.** When the shape of a parabola is described, it is often compared to the graph of $y = x^2$, using terms such as "vertically stretched" (narrower) or "vertically compressed" (wider). Write an equation for a parabola that is

**a)** vertically stretched and opens upward

**b)** vertically stretched and opens downward

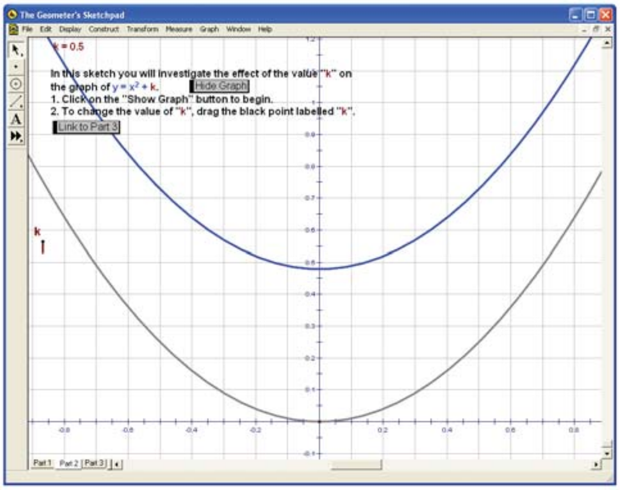
**c)** vertically compressed and opens upward

**d)** vertically compressed and opens downward

## Part B: The Effect of Changing *k*

Select **Link to Part 2**.

**6.** Select **Show Graph** and drag point $k$. How does changing the value of $k$ affect the position of the blue parabola?




**7.** **Reflect** Describe the position of the blue parabola relative to the $x$-axis when $k$ is

**a)** positive

**b)** negative

**c)** zero

**8.** One way to describe the position of the blue parabola is to say that its vertex is translated upward or translated downward relative to the $x$-axis. The equation for the blue parabola is $y = x^2 + k$. Write an equation for a parabola with a vertex

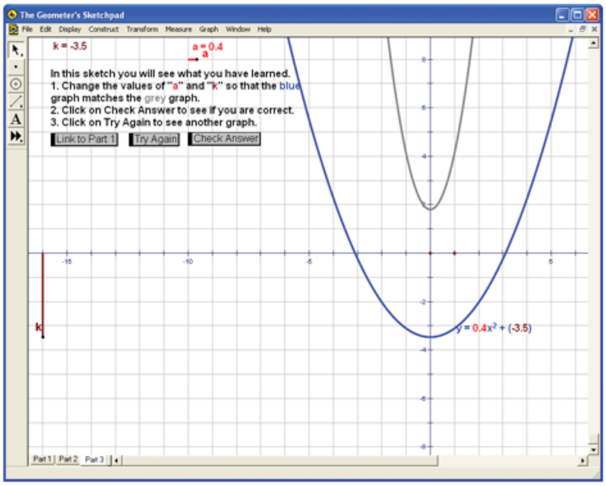
**a)** translated above the $x$-axis

**b)** translated below the $x$-axis

**c)** on the $x$-axis

### Part C: The Effects of Changing *a* and *k*

The effects of changing the values of *a* and *k* can be combined in a new relation $y = ax^2 + k$.




**9.** Select **Link to Part 3**. Find the equation for the grey parabola by changing the values of *a* and *k* in the blue parabola until it matches the grey parabola. Select **Try Again** to input different values of *a* and *k*. When you are satisfied with your solution, select **Check Answer** to see the equation for the grey parabola. A solution that is within one or two tenths is considered correct.

**10.** **Reflect** When matching the blue parabola to the grey parabola, which value was easier to determine, *a* or *k*? Explain.

## Example 1

### Identify Transformations of a Parabola

Describe the transformations that would be applied to the graph of $y = x^2$ to obtain the graph of each relation. Identify the vertex of the new graph. Sketch the graph.

**a)** $y = 3x^2 - 5$

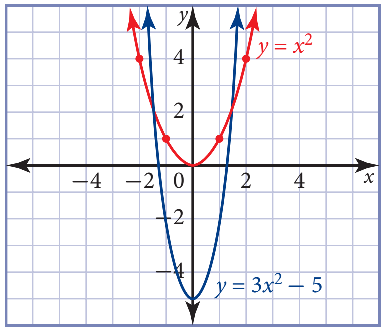
**b)** $y = 0.4x^2 - 10$

**c)** $y = -11x^2 + 8$

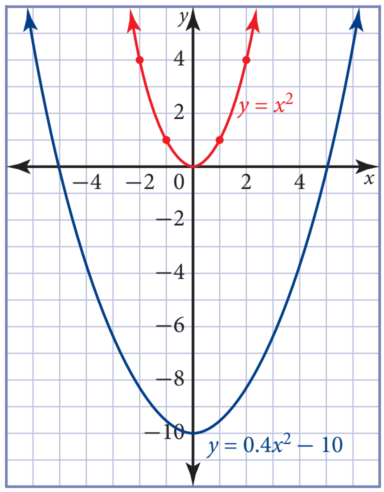
## Solution

Each time, sketch the graph of $y = x^2$. Plot the points (0, 0), (1, 1), (−1, 1), (2, 4), and (−2, 4) and draw a smooth curve through the points. To sketch the graph of the transformed equation, find the coordinates of the vertex for the graph of the new equation. Then adjust the shape of $y = x^2$ if necessary, starting at the new vertex.

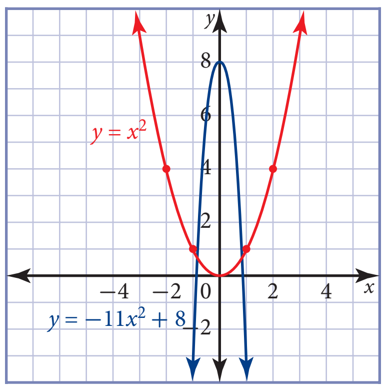
**a)** The parabola is translated 5 units downward, so the vertex is at (0, −5). Since $a > 1$, the parabola opens upward and is vertically stretched.




**b)** The parabola is translated 10 units downward, so the vertex is at (0, −10). Since $0 < a < 1$, the parabola opens upward and is vertically compressed.




**c)** The parabola is translated 8 units upward, so the vertex is at (0, 8). Since $a < -1$, the parabola opens downward and is vertically stretched.

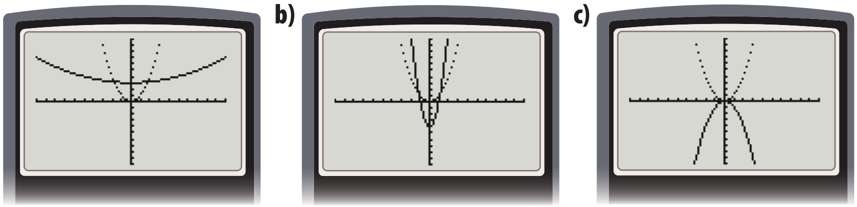


## Example 2

### Describe the Shape of a Parabola

In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola. Describe the shape and position of each solid parabola relative to the graph of $y = x^2$ in terms of $a$ and $k$.

a) b) c)



### Solution

For each parabola, start with the vertex.

**a)** The vertex has been translated 3 units above the $x$-axis, so $k = 3$. The coordinates of the vertex are (0, 3).

The parabola opens upward, so $a$ is positive.

The graph is vertically compressed relative to the graph of $y = x^2$, so $0 < a < 1$.

**b)** The vertex has been translated 4 units below the $x$-axis, so $k = -4$. The coordinates of the vertex are (0, −4).

The parabola opens upward, so $a$ is positive.

The graph is vertically stretched relative to the graph of $y = x^2$, so $a > 1$.

**c)** The vertex is on the $x$-axis, so $k = 0$. The coordinates of the vertex are (0, 0).

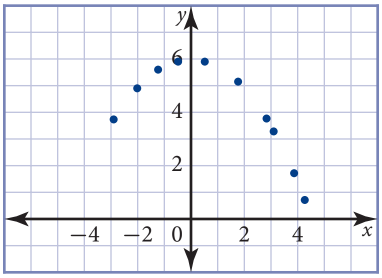
The parabola has been reflected in the $x$-axis.
The parabola opens downward, so $a$ is negative.

The parabola appears to have the same shape as the graph of $y = x^2$, so $a$ is approximately −1.

## Example 3 Predict the Equation of a Parabola Without Using Technology

During filming of the *Lord of the Rings* trilogy, Andy Serkis, the actor who played the character Gollum, wore a motion capture suit. The motion of the dots on the suit was analysed by computer to model the motion of the computer-generated image of Gollum.

Suppose it makes sense to have negative time values. The graph shows Gollum's height above the ground over time as he jumped from one rock to another.



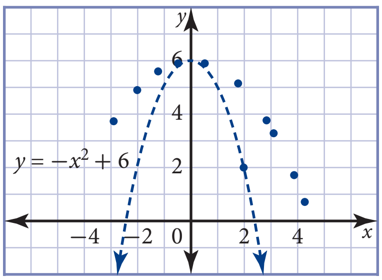

Write a relation in the form $y = ax^2 + k$ to represent Gollum's jump.

### Solution

The vertex is 6 units above the $x$-axis, so $k = 6$. The coordinates of the vertex are (0, 6).

The parabola is reflected in the $x$-axis. The parabola opens downward, so $a$ is negative.

To estimate the value of $a$, graph $y = -x^2 + 6$ on the same set of axes as the parabola that represents Gollum's jump. Here $a = -1$, so the orientations of the parabolas are the same.



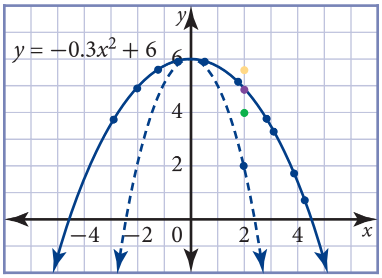

The parabola that represents Gollum's jump is vertically compressed relative to the graph of $y = -x^2 + 6$, so $-1 < a < 0$.

Choosing $a = 0.5$ is a good way to start since it is halfway between 0 and 1.

Try a value greater than $-0.5$, but less than $-0.1$.

Use systematic trial to find the value of $a$. Test a point, such as $x = 2$.

| Value of $a$ | Equation | Calculation | Point | Does it match Gollum's graph? |
|---|---|---|---|---|
| $a = -0.5$<br>$x = 2$ | $y = -0.5x^2 + 6$ | $y = -0.5(2)^2 + 6$<br>$= -2 + 6$<br>$= 4$ | (2, 4) | The point is below the parabola that represents Gollum's jump. The graph of $y = -0.5x^2 + 6$ needs to be vertically compressed. |
| $a = -0.1$<br>$x = 2$ | $y = -0.1x^2 + 6$ | $y = -0.1(2)^2 + 6$<br>$= -0.4 + 6$<br>$= 5.6$ | (2, 5.6) | The point is above the parabola that represents Gollum's jump. The graph of $y = -0.1x^2 + 6$ needs to be vertically stretched. |
| $a = -0.3$<br>$x = 2$ | $y = -0.3x^2 + 6$ | $y = -0.3(2)^2 + 6$<br>$= -1.2 + 6$<br>$= 4.8$ | (2, 4.8) | The point appears to lie on the parabola that represents Gollum's jump. |




Gollum's jump can be modelled by the relation $y = -0.3x^2 + 6$.

## Key Concepts

For any quadratic relation of the form $y = ax^2 + k$:

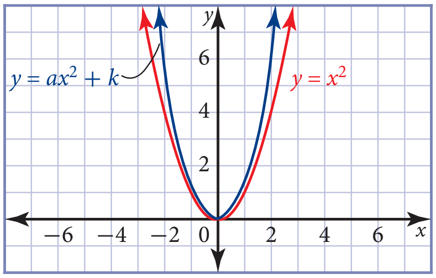
- The value of $a$ determines the orientation and shape of the parabola.
  - If $a > 0$, the parabola opens upward.
  - If $a < 0$, the parabola is reflected in the $x$-axis; it opens downward.
  - If $-1 < a < 1$, the parabola is vertically compressed relative to the graph of $y = x^2$.
  - If $a > 1$ or $a < -1$, the parabola is vertically stretched relative to the graph of $y = x^2$.
- The value of $k$ determines the vertical position of the parabola.
  - If $k > 0$, the vertex of the parabola is $k$ units above the $x$-axis.
  - If $k < 0$, the vertex of the parabola is $k$ units below the $x$-axis.
- The coordinates of the vertex are $(0, k)$.

## Discuss the Concepts

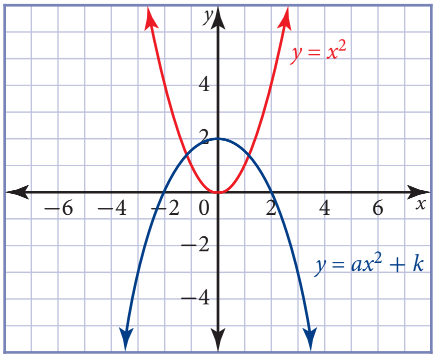
**D1.** You are generating a table of values for a relation of the form $y = ax^2 + k$. Why are the coordinates of the vertex the easiest points to determine?

**D2.** Compare each parabola labelled $y = ax^2 + k$ to the graph of $y = x^2$. Describe what you know about the values of $a$ and $k$.

**a)**




**b)**

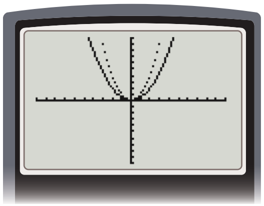



## Practise A

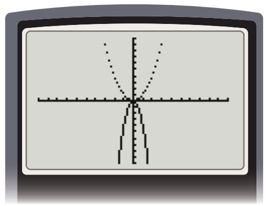
**For help with questions 1 to 4, refer to Example 1.**

**1.** In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola and the graph of a relation of the form $y = ax^2$ is shown as a solid parabola. For each solid parabola, is $a$ less than $-1$, between 0 and $-1$, between 0 and 1, or greater than 1? Explain.

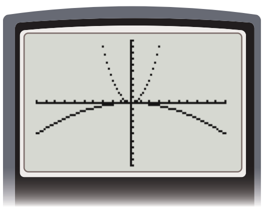
**a)**



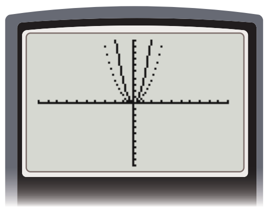
**b)**



**c)**

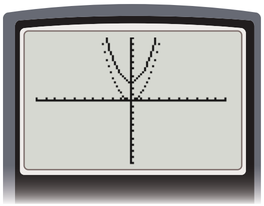


**d)**

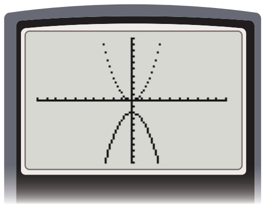


**2.** In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola and the graph of a relation of the form $y = ax^2 + k$ is shown as a solid parabola. For each solid parabola, is $k$ positive or negative? Explain.

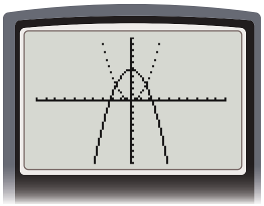
**a)**



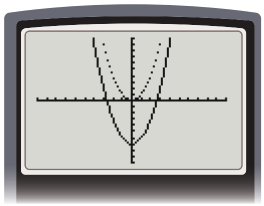
**b)**



**c)**



**d)**



**3.** For each solid parabola in question 2, identify the value of $k$ and the coordinates of the vertex.

**4.** Describe the shape and position of each parabola relative to the graph of $y = x^2$. Sketch each graph.

**a)** $y = 3x^2$

**b)** $y = x^2 + 3$

**c)** $y = -0.5x^2$

**d)** $y = x^2 - 12$

**e)** $y = 0.15x^2 + 13$

**f)** $y = -7x^2 + 6$

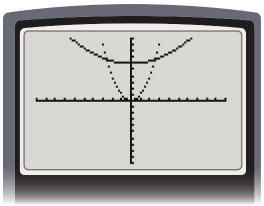
**g)** $y = -0.3x^2 - 5$

**h)** $y = 10x^2 - 9$

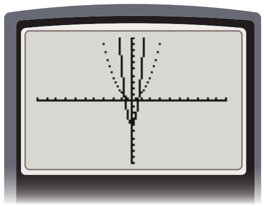
**For help with question 5, refer to Example 2.**

**5.** In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola. Describe the shape and position of each solid parabola relative to the graph of $y = x^2$ in terms of $a$ and $k$.

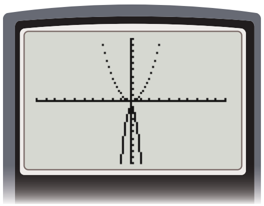
**a)**



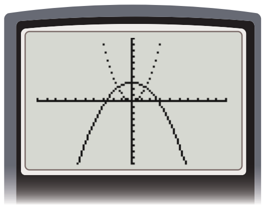
**b)**



**c)**



**d)**



**For help with question 6, refer to Example 3.**

**6.** Graph each relation. Then, represent the relation with an equation of the form $y = ax^2 + k$.

**a)**

| x | y |
|---|---|
| −0.5 | −5.0 |
| 0.0 | −6.0 |
| 0.5 | −5.0 |
| 1.0 | −2.0 |
| 1.5 | 3.0 |
| 2.0 | 10.0 |

**b)**

| x | y |
|---|---|
| −10 | 1 |
| −5 | 4 |
| 0 | 5 |
| 5 | 4 |
| 10 | 1 |
| 15 | −4 |

**c)**

| x | y |
|---|---|
| −9 | −48 |
| −6 | −23 |
| −3 | −8 |
| 0 | −3 |
| 3 | −8 |
| 6 | −23 |

## Apply B

**7.** Suppose each pair of relations were graphed on the same set of axes.

- Which parabola would be the widest (most vertically compressed)?
- Which parabola would have its vertex farther from the $x$-axis?

Justify your answers.

**a)** $y = 0.2x^2$ $\quad$ $y = 5x^2 + 6$

**b)** $y = 3x^2 + 9$ $\quad$ $y = -0.4x^2 - 8$

**c)** $y = 5x^2 + 7$ $\quad$ $y = 2x^2 - 5$

**d)** $y = 0.1x^2$ $\quad$ $y = 0.25x^2 + 11$

**e)** $y = -0.2x^2 - 1$ $\quad$ $y = 0.03x^2 + 2$

**f)** $y = x^2 - 6$ $\quad$ $y = 0.9x^2 + 6$

**Chapter Problem**

**8.** You are creating a computer model of a skateboarder jumping off a ramp for the next part of your game. The path of the skateboarder is modelled by the relation $h = -0.85t^2 + 2$, where $h$ is the skateboarder's height above the ground, in metres, and $t$ is the time, in seconds.

**a)** How far from the ground will the skateboarder be after 0.5 s? After 1 s?

**b)** For how long will the skateboarder be in the air?

**9.** The stopping distance of a particular car can be modelled by the relation $d = 0.006s^2$, where $d$ is the distance, in metres, and $s$ is the speed in kilometres per hour.

**a)** Graph the relation.

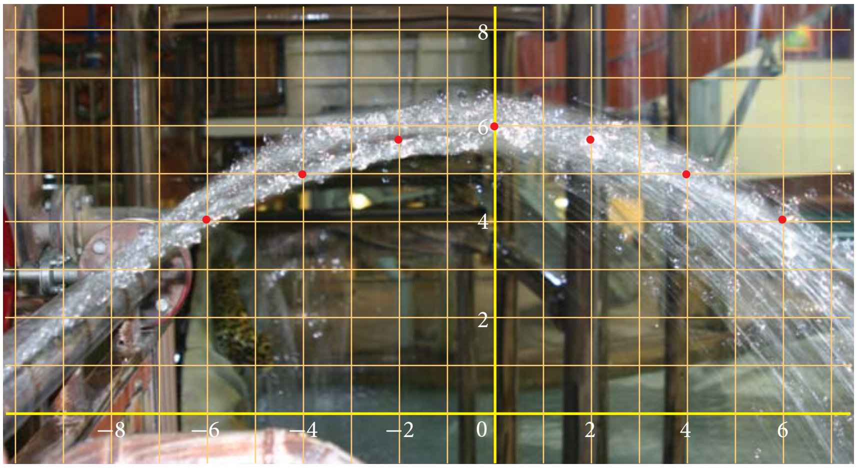
**b)** Determine the stopping distance for the car if it is travelling at

**i)** 50 km/h **ii)** 60 km/h **iii)** 100 km/h **iv)** 110 km/h

**Literacy Connect**

c) The speed limit on city roads is 50 km/h, while the speed limit on highways is 100 km/h. Use your answers to part b) to calculate the extra stopping distance needed for a car travelling 10 km/h over the speed limit in the city. Compare this to the extra stopping distance needed for a car travelling 10 km/h over the speed limit on the highway.

**10.** The path of water from this fountain can be modelled by a relation of the form $y = ax^2 + k$.




a) What are the coordinates of the vertex of this parabola?

b) Determine the approximate value of $a$.

c) Write an equation to model the parabola.

## Extend C

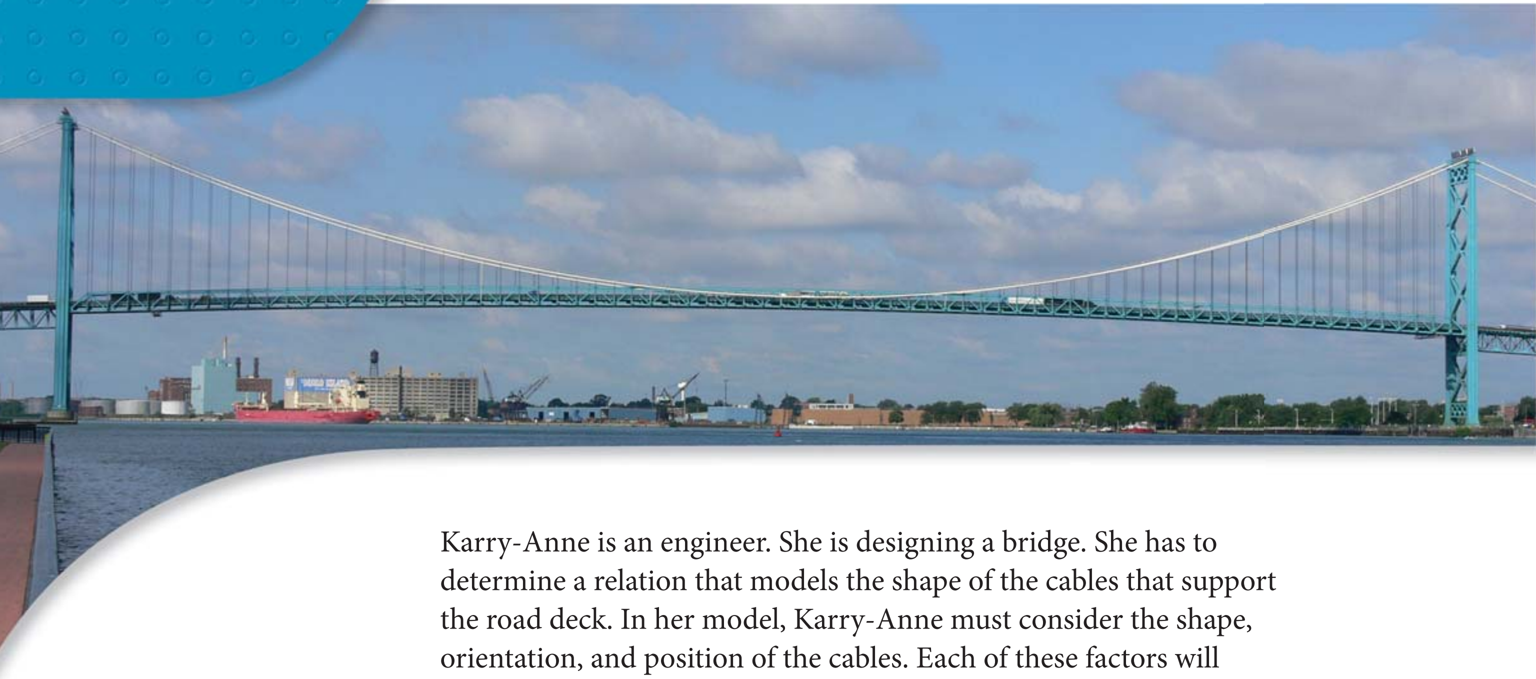
**11.** In January 2006, Jamie Pierre set a new world record with a cliff jump of more than 77 m into deep snow. His jump from the top of the cliff can be modelled by the relation $d = 4.9t^2$, where $d$ is his distance from below the top of the cliff, in metres, and $t$ is the time, in seconds.

a) How much farther did Jamie fall in the third second of his jump compared to the first second?

b) How long did it take Jamie to make the 77-m jump?

**12.** The path of a cannonball shot horizontally off a cliff can be modelled by the relation $h = 25 - \frac{4.9d^2}{{V_0}^2}$, where $h$ is the cannonball's height above the ground, in metres, $d$ is the cannonball's horizontal distance from the base of the cliff, in metres, and $V_0$ is the initial velocity of the cannonball, in metres per second. How much farther will a cannonball travel before hitting the ground if it is shot at an initial velocity of 50 m/s compared to an initial velocity of 35 m/s?

# 4.3 The Quadratic Relation $y = a(x - h)^2$



Karry-Anne is an engineer. She is designing a bridge. She has to determine a relation that models the shape of the cables that support the road deck. In her model, Karry-Anne must consider the shape, orientation, and position of the cables. Each of these factors will contribute to the formula, as you will learn in your investigation of graphs in this section.

## Investigate

### Graphs of $y = a(x - h)^2$

**Tools**

- graphing calculator

**Method 1: Use a Graphing Calculator**

Clear all equations from the equation window.

**1.** Start by graphing $y = x^2$.

- Press $\boxed{\text{Y=}}$ and enter $x^2$ for Y1.
- Use the left arrow key to move the cursor as far left as it will go. Press $\boxed{\text{ENTER}}$ repeatedly to change the line style to the dotted line.
- Press $\boxed{\text{ZOOM}}$ **6:ZStandard**.
- As you add new relations, keep the graph of $y = x^2$ for comparison.

**Part A: The Effect of Changing *h***

**2.** **a)** Graph each relation.

**i)** $y = (x + 2)^2$ **ii)** $y = (x + 4)^2$

**b)** How does the shape of each graph compare to the shape of $y = x^2$?

**c)** What is similar about the positions of all three graphs?

**d)** Each relation in part a) has a constant term inside the brackets. How does this constant term relate to the position of the graph?

3. a) Graph each relation.

   i) $y = (x - 3)^2$ ii) $y = (x - 6)^2$

   b) What is similar about the positions of these graphs and the position of $y = x^2$?

   c) Each relation in part a) has a constant term inside the brackets. How does the constant term relate to the position of the graph?

4. A quadratic relation is in the form $y = (x - h)^2$. Describe the position of the graph of $y = (x - h)^2$ relative to the $y$-axis when $h$ is

   a) positive b) negative c) zero

5. A quadratic relation is in the form $y = (x - h)^2$. Substitute a value for $h$ to write an equation for a parabola with a vertex

   a) translated to the right of the $y$-axis

   b) translated to the left of the $y$-axis

   c) on the $y$-axis

### Part B: The Effects of Changing *a* and *h*

The effects of $a$ and $h$ can be combined in a new relation $y = a(x - h)^2$.

6. Write an equation for each parabola. Use a graphing calculator to check your answers.

   a) vertically compressed, opens upward, vertex at (5, 0)

   b) vertically stretched, opens downward, vertex is (−3, 0)

   c) vertically stretched, opens upward, vertex is on the $y$-axis

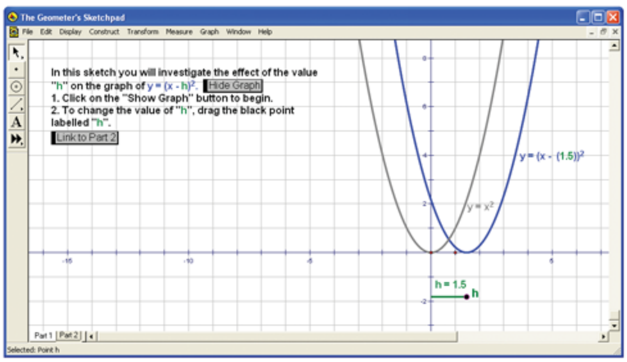
7. **Reflect** Suppose you are to write a relation in the form $y = a(x - h)^2$ given the graph of a parabola. Which value is easier to determine, $a$ or $h$? Explain.

**Tools**

- computers
- *The Geometer's Sketchpad®*
- 4.3 Investigation.gsp

### Method 2: Use *The Geometer's Sketchpad®*

Go to *www.mcgrawhill.ca/links/foundations11* and follow the links to 4.3. Download the file **4.3 Investigation.gsp**. Open the sketch.



**Part A: The Effect of Changing *h***

1. Select **Show Graph** and drag point $h$. How does changing the value of $h$ affect the position of the blue parabola?

2. Describe the position of the blue parabola relative to the $y$-axis when $h$ is

   **a)** positive  **b)** negative  **c)** zero

3. The blue parabola can be represented by a relation of the form $y = (x - h)^2$. Substitute a value for $h$ to write an equation for a parabola with a vertex

   **a)** translated to the right of the $y$-axis

   **b)** translated to the left of the $y$-axis

   **c)** on the $y$-axis

**Part B: The Effects of Changing *a* and *h***

The effects of $a$ and $h$ can be combined in a new relation $y = a(x - h)^2$.

4. Select **Link to Part 2**. Find an equation for the grey parabola by changing the values of $a$ and $h$ in the blue parabola until it matches the grey parabola. Select **Try Again** to input different values of $a$ and $h$. When you are satisfied with your solution, select **Check Answer** to see the equation for the grey parabola. A solution that is within one or two tenths is considered correct.

5. **Reflect** You are given the graph of a parabola and must write the equation of the parabola in the form $y = a(x - h)^2$. Which value is easier to determine, $a$ or $h$? Explain.

Notice the negative sign in the equation $y = a(x - h)^2$.

When $h$ is positive, the number after the subtraction symbol is positive.

For example,

$y = 3(x - \boxed{8})^2$  $y = a(x - \boxed{h})^2$  The 8 replaces the $h$ and the subtraction symbol does not change.

$h = +8$

When $h$ is negative, the number after the subtraction symbol is negative.

For example,

$y = 4(x + 6)^2$ becomes

$y = 4(x - (\boxed{-6}))^2$  $y = a(x - \boxed{h})^2$

$h = -6$

## Example 1

### Identify Transformations of a Parabola

Describe the transformations that would be applied to the graph of $y = x^2$ to obtain the graph of each relation. Identify the vertex of the new graph. Sketch the graph.

**a)** $y = -0.1(x - 6)^2$

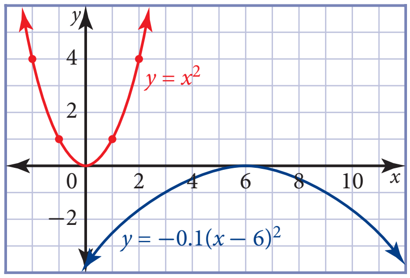
**b)** $y = 4(x + 7)^2$

**c)** $y = 0.9(x + 3)^2$

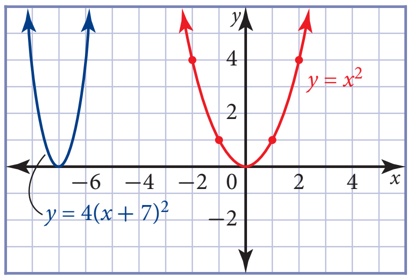
### Solution

Each time, sketch the graph of $y = x^2$. Plot the points (0, 0), (1, 1), (−1, 1), (2, 4), and (−2, 4) and draw a smooth curve through the points. To sketch the graph of the transformed equation, find the coordinates of the vertex for the graph of the new equation. Then adjust the shape of $y = x^2$ if necessary.

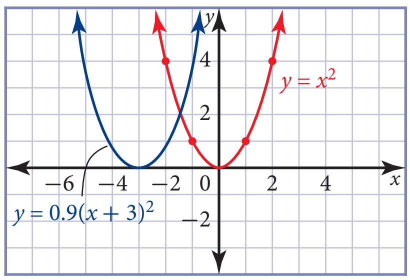
**a)** The parabola is translated 6 units to the right, so the vertex is at (6, 0).
Since $-1 < a < 0$, the parabola is reflected in the $x$-axis (it opens downward) and is vertically compressed.




**b)** The parabola is translated 7 units to the left, so the vertex is at (−7, 0).
Since $a > 1$, the parabola is vertically stretched and opens upward.



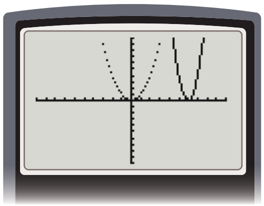
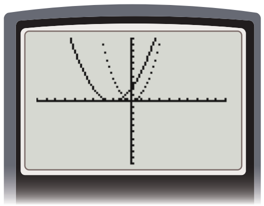
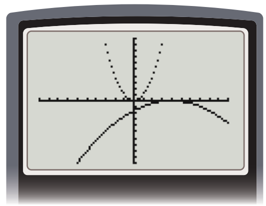

**c)** The parabola is translated 3 units to the left, so the vertex is at (−3, 0).
Since $0 < a < 1$, the parabola is vertically compressed and opens upward.



**Example 2**

## Describe the Shape of a Parabola

In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola. Describe the shape and position of each solid parabola relative to the graph of $y = x^2$ in terms of $a$ and $h$.

**a)**

**b)**

**c)**


### Solution

For each graph, start with the vertex.

**a)** The vertex has been translated 6 units to the right of the $y$-axis, so $h = 6$. The coordinates of the vertex are (6, 0).

The parabola opens upward, so $a$ is positive.

The parabola is vertically stretched relative to the graph of $y = x^2$, so $a > 1$.

**b)** The vertex has been translated 2 units to the left of the $y$-axis, so $h = -2$. The coordinates of the vertex are $(-2, 0)$.

The parabola opens upward, so $a$ is positive.

The parabola is vertically compressed relative to the graph of $y = x^2$, so $0 < a < 1$.

**c)** The vertex has been translated 4 units to the right of the $y$-axis, so $h = 4$. The coordinates of the vertex are (4, 0).

The parabola has been reflected in the $x$-axis. The parabola opens downward, so $a$ is negative.

The parabola is vertically compressed relative to the graph of $y = x^2$, so $-1 < a < 0$.

## Key Concepts

For any quadratic relation of the form $y = a(x - h)^2$:

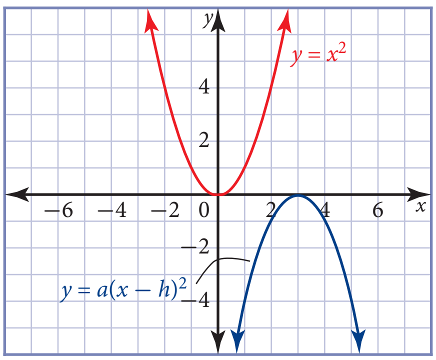
- The value of $a$ determines the orientation and shape of the parabola.
  - If $a > 0$, the parabola opens upward.
  - If $a < 0$, the parabola opens downward.
  - If $-1 < a < 1$, the parabola is vertically compressed relative to the graph of $y = x^2$.
  - If $a > 1$ or $a < -1$, the parabola is vertically stretched relative to the graph of $y = x^2$.
- The value of $h$ determines the horizontal position of the parabola.
  - If $h > 0$, the vertex of the parabola is $h$ units to the right of the $y$-axis.
  - If $h < 0$, the vertex of the parabola is $h$ units to the left of the $y$-axis.
- The coordinates of the vertex are $(h, 0)$.

### Discussion the Concepts

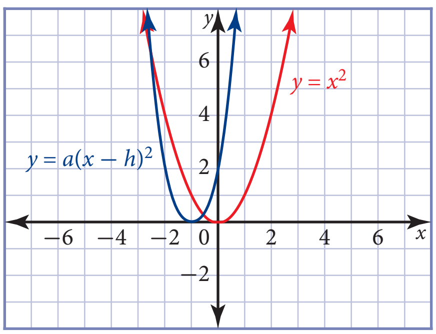
**D1.** You are given a graph of a parabola and must write its equation in the form $y = a(x - h)^2$. Which value is easier to determine, $a$ or $h$? Explain.

**D2.** Compare each parabola labelled $y = a(x - h)^2$ to the graph of $y = x^2$. Describe what you know about the values of $a$ and $h$.

**a)**




**b)**

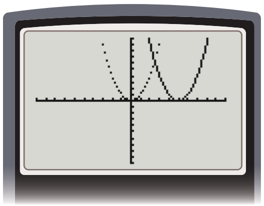


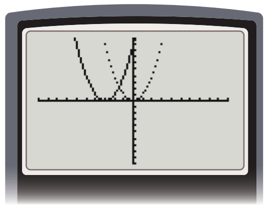
## Practise A

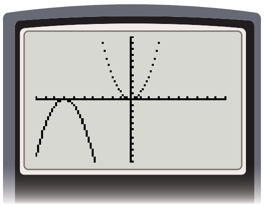
**For help with questions 1 and 2, refer to Example 1.**

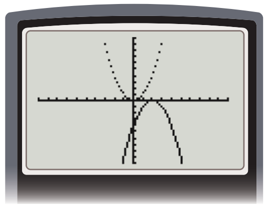
**1.** In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola and the graph of a relation of the form $y = a(x - h)^2$ is shown as a solid parabola.

For each solid parabola, identify the value of $h$ and the coordinates of the vertex.

**a)**


**b)**


**c)**


**d)**


**2.** Describe the graph of each parabola relative to the graph of $y = x^2$ in terms of $a$ and $h$. Sketch each graph.

**a)** $y = (x - 7)^2$

**b)** $y = -(x + 3)^2$

**c)** $y = 1.5(x + 8)^2$

**d)** $y = -0.8(x - 2)^2$

**e)** $y = 0.1(x - 5)^2$

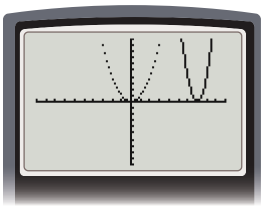
**f)** $y = 2(x + 1)^2$

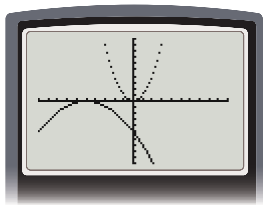
**g)** $y = -2(x - 8)^2$

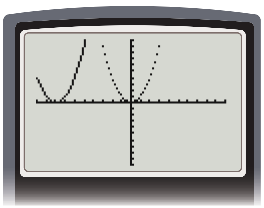
**h)** $y = 0.3(x + 14)^2$

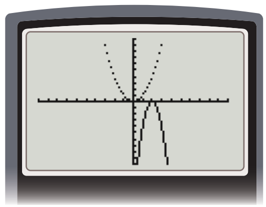
**For help with question 3, refer to Example 2.**

**3.** In each standard viewing window, the graph of $y = x^2$ is shown as a dotted parabola. Describe the shape and position of each solid parabola relative to the graph of $y = x^2$ in terms of $a$ and $h$.

**a)**


**b)**


**c)**


**d)**


4. Graph each relation. Then, write a relation in the form of $y = a(x - h)^2$ that models each parabola.

**a)**

| x | y |
|---|---|
| 0 | −32 |
| 1 | −18 |
| 2 | −8 |
| 3 | −2 |
| 4 | 0 |
| 5 | −2 |

**b)**

| x | y |
|---|---|
| −7.0 | 2.0 |
| −6.0 | 0.5 |
| −5.0 | 0.0 |
| −4.0 | 0.5 |
| −3.0 | 2.0 |
| −2.0 | 4.5 |

## Apply B

5. Suppose each pair of relations were graphed on one set of axes. Which parabola would have its vertex farther from the $y$-axis? Justify your answers.

**a)** $y = 2(x + 3)^2$ $\quad$ $y = 0.1(x + 1)^2$

**b)** $y = 9(x - 3)^2$ $\quad$ $y = -0.2(x - 8)^2$

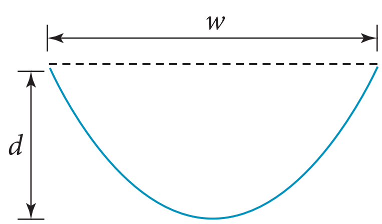
**c)** $y = 0.001(x + 3)^2$ $\quad$ $y = 32(x - 10)^2$

**d)** $y = -15(x - 2)^2$ $\quad$ $y = 0.85(x + 9)^2$

**Literacy Connect**

6. Why do you think the general equation for a quadratic relation is written as $y = a(x - h)^2$ instead of $y = a(x + h)^2$?


7. The side view of a car headlight shows that the back part of the headlight is parabolic. For the headlight shown, the shape of the parabolic part can be modelled by the relation $d = 0.08(w - 7)^2$, where $d$ is the depth of the parabola and $w$ is the width of the headlight from edge to edge.




**a)** Draw a graph to represent the shape of the back part of the headlight.

**b)** How wide is the headlight when the parabola is 4 cm deep?

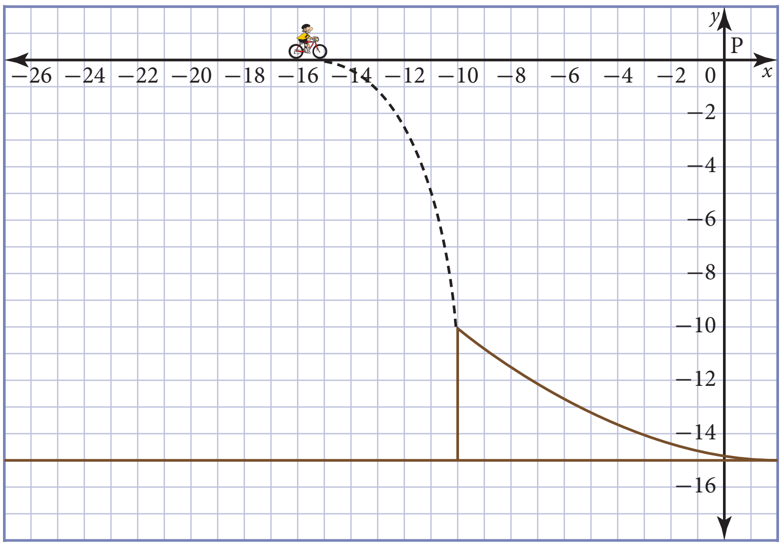
**Chapter Problem**

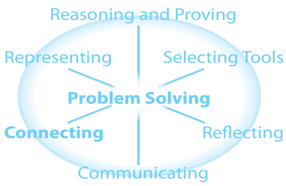
**8.** In another part of the video game, you are modelling a mountain biker performing a stunt. The mountain biker is speeding along the edge of a mesa toward a cliff, marked *P*. A rock ramp begins 10 m before, and 10 m below, *P*. The mountain biker must judge when to jump off the mesa to land safely on the ramp. The mountain biker's path through the air (shown with the dotted line) can be modelled using the relation $y = -\frac{5}{v^2}(x - h)^2$, where $v$ is the mountain biker's speed, in metres per second, $y$ is the mountain biker's height above the top of the mesa, in metres, and $x$ is the mountain biker's horizontal distance from the cliff, *P*, in metres.

**a)** The value of $h$ is the distance from the cliff at which the mountain biker must jump to land safely on the ramp, for a given speed. Suppose the mountain biker's speed is 5 m/s. Find the value of $h$.

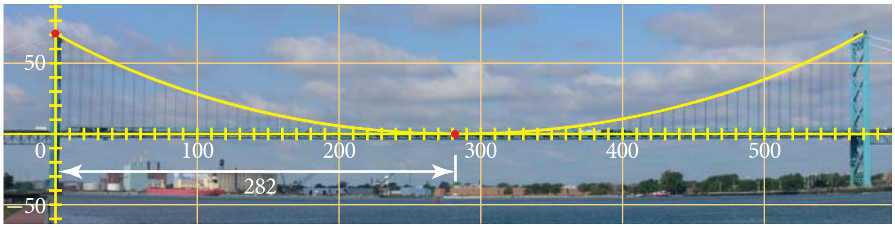
**b)** If the mountain biker's speed increases, will the value of $h$ from part a) increase, decrease, or remain the same? Explain your reasoning.

**c)** Check your answer to part b) by calculating a value of $h$ for a speed of 10 m/s.






**9.** The curve formed by the cables on a suspension bridge can be modelled by the relation $y = a(x - h)^2$, where $y$ is the height above the bridge deck and $x$ is the horizontal distance from one support tower, both in metres.




**a)** What are the coordinates of the vertex of this parabola?

**b)** Determine the approximate value of $a$.

**c)** Write an equation for the quadratic relation that models the parabola.

**d)** Based on your equation, at what height above the deck are the cables attached to the support tower?

## Extend C

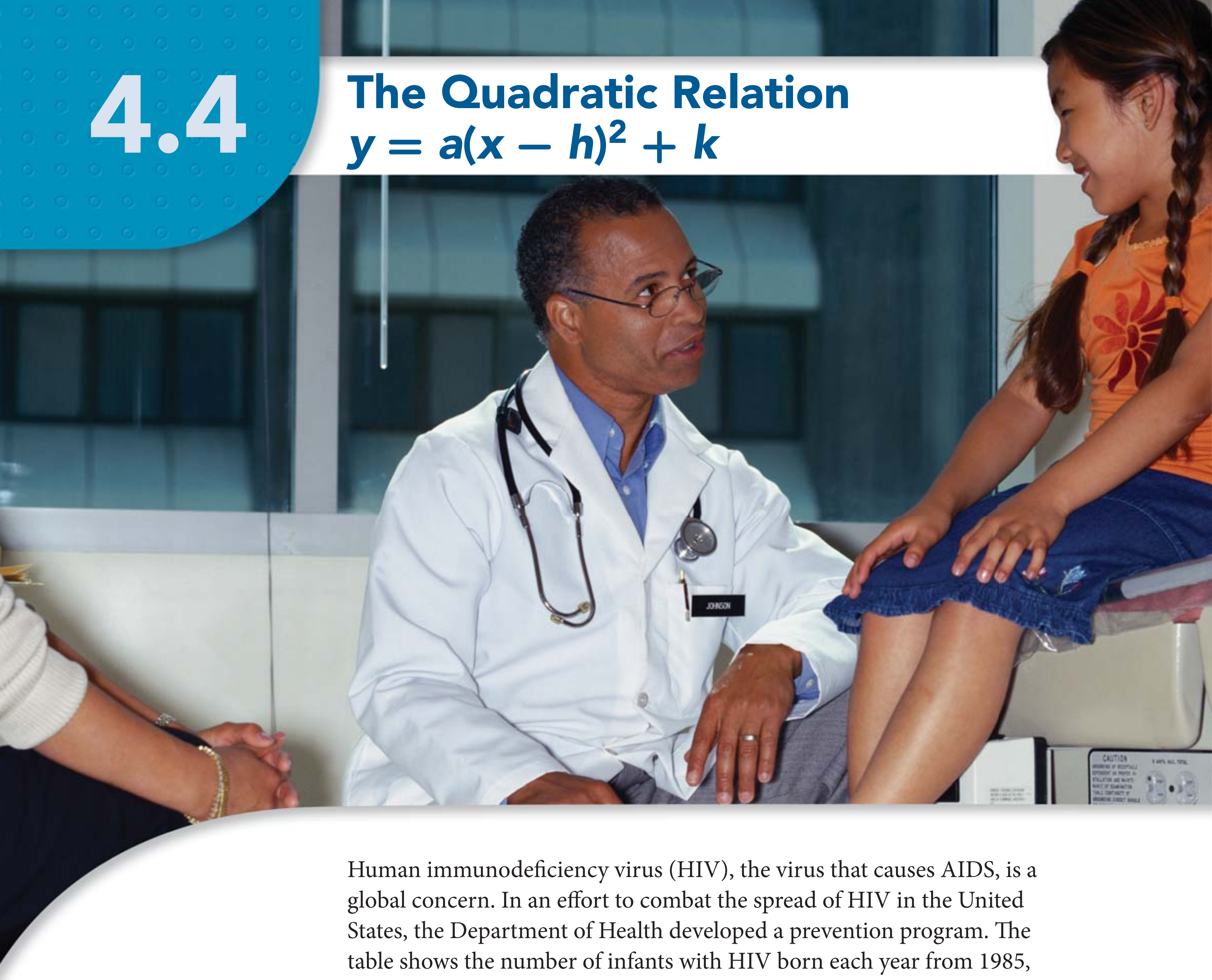
**10.** **a)** Copy and complete the table.

| $x$ | $y = (x - 3)^2$ | $y = 3 + \sqrt{x}$ | $y = 3 - \sqrt{x}$ |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |

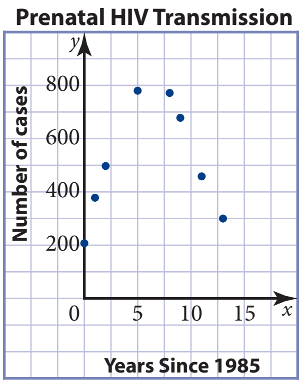
**b)** Graph the three relations on the same set of axes.

**c)** Compare the second and third graphs to the first graph. How are the graphs similar? How are they different?

# 4.4 The Quadratic Relation $y = a(x - h)^2 + k$




Human immunodeficiency virus (HIV), the virus that causes AIDS, is a global concern. In an effort to combat the spread of HIV in the United States, the Department of Health developed a prevention program. The table shows the number of infants with HIV born each year from 1985, the year the program began, to 1998.




| Year | Years Since 1985 | Number of Cases |
|---|---|---|
| 1985 | 0 | 210 |
| 1986 | 1 | 380 |
| 1987 | 2 | 500 |
| 1990 | 5 | 780 |
| 1993 | 8 | 770 |
| 1994 | 9 | 680 |
| 1996 | 11 | 460 |
| 1998 | 13 | 300 |

The relationship between the number of infants with HIV and the number of years since the start of the program appears to be quadratic.

## Investigate — Graphs of $y = a(x - h)^2 + k$

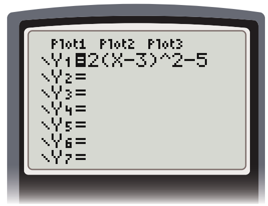
### Tools

- graphing calculator

**Method 1: Use a Graphing Calculator**

Clear all equations from the [Y=] window and turn off any stat plots.

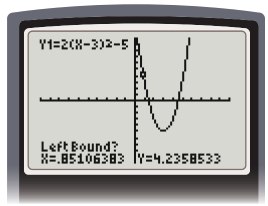
**1.** Start by graphing $y = 2(x - 3)^2 - 5$.
- Press [Y=] and enter $2(x - 3)^2 - 5$ for Y1.
- Press [ZOOM] **6:ZStandard** to graph the relation in the standard window.




**Technology Tip**

If you do not see the cursor, it might be off screen. Look at the coordinates at the bottom of the screen to locate the cursor. Use the arrow keys to move the cursor into the visible screen area.

**2.** Find the minimum value for the graph.
- Press [2nd] [CALC] **3:minimum**.
- Use the arrow keys to move the cursor to the left of the minimum point and press [ENTER].
- Use the arrow keys to move the cursor to the right of the minimum point and press [ENTER] twice.
- The minimum value will be displayed. Record the equation and the coordinates of the vertex.




**3.** Graph each relation. Find the minimum value for each graph. Record each equation and the coordinates of the vertex.

**a)** $y = 0.2(x - 5)^2 + 7$

**b)** $y = 10(x - 8)^2 - 9$

**4.** For these relations, $a$ is negative, so the $y$-coordinate of the vertex represents a maximum. Graph each equation. Find the maximum value for each graph. Record each relation and the coordinates of the vertex.

**a)** $y = -4(x + 7)^2 + 2$

**b)** $y = -0.5(x - 8)^2 + 1$

**c)** $y = -9(x + 4)^2 - 6$

**5.** Refer to your answers to questions 2 to 4. For each set of vertex coordinates, how does the minimum or maximum value relate to the relation?

**6.** **Reflect** If you are given the coordinates of the vertex of a parabola, do you have enough information to write an equation for the parabola? Explain why or why not.

**Tools**

- computer
- *The Geometer's Sketchpad®*
- 4.4 Investigation.gsp

**Method 2: Use *The Geometer's Sketchpad®*.**

Go to *www.mcgrawhill.ca/links/foundations11* and follow the links to 4.4. Download the file **4.4 Investigation.gsp**. Open the sketch.

1. Change the values of $a$, $k$, and $h$ to create new parabolas. Which of the coefficients relate directly to the coordinates of the vertex?

2. Record the equation and the coordinates of the vertex for three different parabolas that
   a) open upward
   b) open downward

3. Refer to your answers to question 2. For each set of vertex coordinates, how does the minimum or maximum value relate to the equation?

   The effects of the values of $a$, $h$, and $k$ can be combined in a new equation, $y = a(x - h)^2 + k$.

4. Find the equation for the grey parabola by changing the values of $a$, $h$, and $k$ in the blue parabola until it matches the grey parabola. Select **Try Again** to input different values of $a$, $h$, and $k$. When you are satisfied with your solution, select **Check Answer** to see the equation for the grey parabola. A solution that is within one or two tenths is considered correct.

5. **Reflect** In question 4, did it matter in which order you changed the values of $a$, $h$, and $k$? Was there an order that helped you find the solution more easily?

**vertex form**

- a quadratic relation of the form $y = a(x - h)^2 + k$
- the coordinates of the vertex are $(h, k)$

The equation $y = a(x - h)^2 + k$ is the **vertex form** of a quadratic relation.

## Example 1

## Sketch Graphs of $y = a(x - h)^2 + k$

Describe the graph of each relation, and identify the coordinates of the vertex. Sketch each graph.

a) $y = 15(x + 5)^2 + 7$

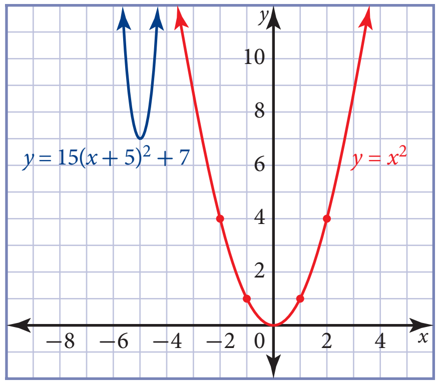
b) $y = 0.5(x - 1)^2 + 9$

c) $y = -4(x + 6)^2 - 8$

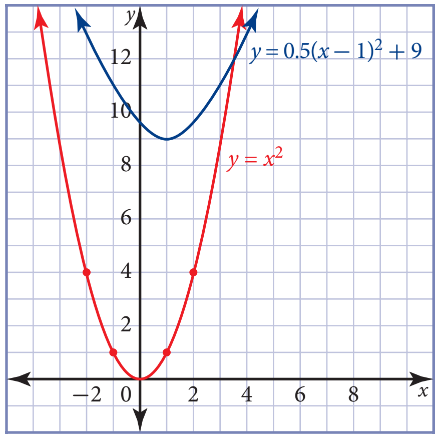
## Solution

To sketch the graph of $y = x^2$, plot the points (1, 1), (−1, 1), (0, 0), (2, 4), and (−2, 4) and draw a smooth curve through the points. To sketch the graph of the transformed equation, find the coordinates of the vertex for the graph of the new equation.

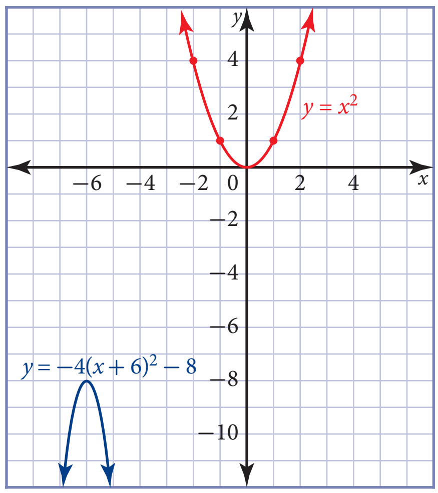
**a)** The coordinates of the vertex are (−5, 7).
Since $a > 1$, the parabola opens upward and is vertically stretched.




**b)** The coordinates of the vertex are (1, 9).
Since $0 < a < 1$, the parabola opens upward and is vertically compressed.




**c)** The coordinates of the vertex are (−6, −8).
Since $a < -1$, the parabola opens downward and is vertically stretched.



## Example 2

### Use Points to Sketch Graphs of $y = a(x - h)^2 + k$

For each relation:

**i)** Identify the coordinates of the vertex.

**ii)** Determine the $x$-coordinate of a point
- 2 units to the left of the vertex
- 2 units to the right of the vertex

**iii)** Use the $x$-values from part ii) to find two points on the parabola. Plot the two points and the vertex on the same set of axes.

**iv)** Sketch the parabola by drawing a smooth curve through the points.

**a)** $y = 2(x - 6)^2 + 3$

**b)** $y = -0.5(x + 9)^2 - 2$

### Solution

**a)** $y = 2(x - 6)^2 + 3$

**i)** vertex: (6, 3)

**ii)** 2 units left: $x = 6 - 2$
$= 4$

2 units right: $x = 6 + 2$
$= 8$

**iii)** When $x = 4$,

$$\begin{aligned} y &= 2(4 - 6)^2 + 3 \\ &= 2(-2)^2 + 3 \\ &= 11 \end{aligned}$$

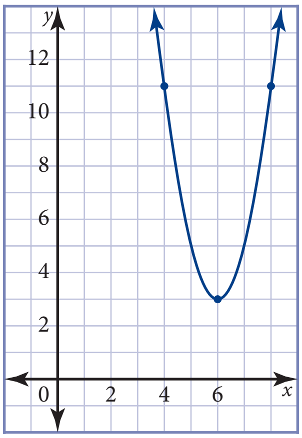
The first point is (4, 11).

When $x = 8$,

$$\begin{aligned} y &= 2(8 - 6)^2 + 3 \\ &= 2(2)^2 + 3 \\ &= 11 \end{aligned}$$

The second point is (8, 11).

**iv)**




Check: $a$ is positive, so the parabola opens upward. $a > 1$, so the parabola is vertically stretched relative to the graph of $y = x^2$.

**b)** 
$$\begin{aligned} y &= -0.5(x + 9)^2 - 2 \\ &= -0.5(x - (-9))^2 + (-2) \end{aligned}$$

**i)** vertex: (−9, −2)

**ii)** 2 units left: $x = -9 - 2$
$= -11$

2 units right: $x = -9 + 2$
$= -7$

**iii)** When $x = -11$,

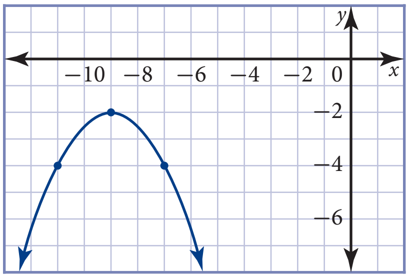
$y = -0.5(-11 + 9)^2 - 2$
$\quad = -0.5(-2)^2 - 2$
$\quad = -4$

When $x = -7$,

$y = -0.5(-7 + 9)^2 - 2$
$\quad = -0.5(2)^2 - 2$
$\quad = -4$

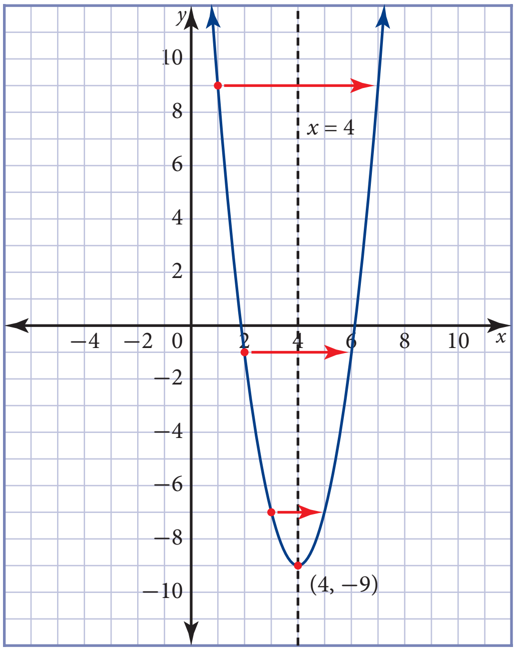
The first point is $(-11, -4)$. The second point is $(-7, -4)$.

**iv)**




Check: $a$ is negative, so the parabola opens downward. $-1 < a < 1$, so the parabola is vertically compressed relative to the graph of $y = x^2$.

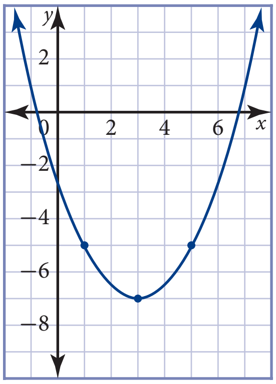
Due to the symmetry of a parabola, only the vertex and one other point are needed to sketch a parabola. For the non-vertex point, there is a corresponding point that is equidistant from the line of symmetry, $x = h$.



## Example 3

### Use Points to Determine the Value of *a*

Consider this parabola.




**a)** Find the coordinates of the vertex, and the values of $h$ and $k$.

**b)** Identify the coordinates of the two other points shown.

**c)** Find the value of $a$ by substituting the coordinates of the vertex and one of the other two points into the relation $y = a(x - h)^2 + k$.

**d)** Write an equation for the parabola.

### Solution

**a)** The vertex is $(3, -7)$, so $h = 3$ and $k = -7$.

**b)** The coordinates of the other two points are $(1, -5)$ and $(5, -5)$.

**c)** Use $(x, y) = (1, -5)$ and $(h, k) = (3, -7)$ to solve for $a$.

$$y = a(x - h)^2 + k$$
$$-5 = a(1 - 3)^2 + (-7)$$
$$-5 = a(-2)^2 - 7$$ Evaluate the expression in brackets, then simplify.
$$-5 = a(4) - 7$$
$$-5 + 7 = 4a - 7 + 7$$ Add 7 to both sides.
$$2 = 4a$$
$$\frac{2}{4} = \frac{4a}{4}$$ Divide both sides by 4.
$$0.5 = a$$

**d)** An equation for the parabola is $y = 0.5(x - 3)^2 - 7$.

## Key Concepts

- For any quadratic relation of the form $y = a(x - h)^2 + k$:
  - The value of $a$ determines the orientation and shape of the parabola relative to the graph of $y = x^2$.
  - The coordinates of the vertex of the parabola are $(h, k)$.
- To sketch a given quadratic relation, plot the vertex and two other points, one on either side of the vertex. Then, draw a smooth curve through the points.
- To determine the equation for a quadratic relation from a graph, use the vertex and another point to solve for $a$. Then, write the relation using the vertex and the value of $a$.

### Discuss the Concepts

**D1.** There are two methods for graphing a quadratic relation:

**A** determine the values of $a$, $h$, and $k$ in the relation $y = a(x - h)^2 + k$ and sketch the graph

**B** determine three points that satisfy the relation, plot the points, and join the points with a smooth curve

**a)** Which method do you think is easier? Why?

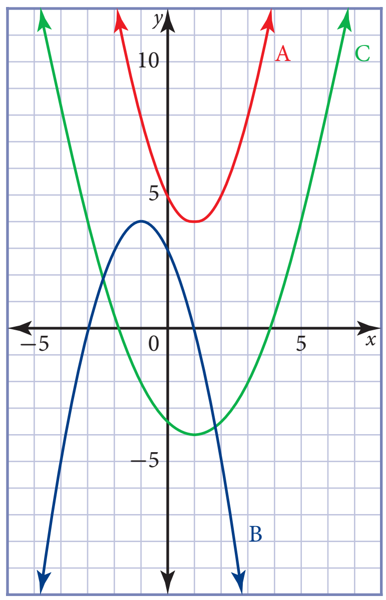
**b)** Which method do you think is more accurate? Why?

**D2.** Match each relation with its graph.

**a)** $y = -(x + 1)^2 + 4$

**b)** $y = 0.5(x - 1)^2 - 4$

**c)** $y = (x - 1)^2 + 4$

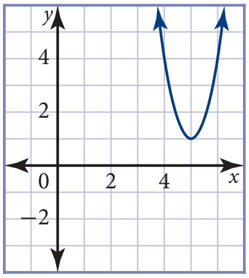


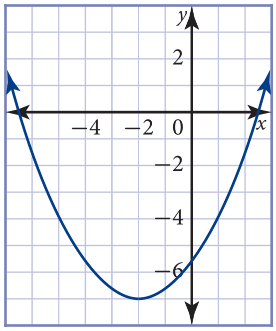

**D3.** Refer to Example 2, part iii). When you calculated the coordinates for the points two units to the right and the left of the vertex, what similarities did you notice in the calculations? How might these change if you found the coordinates of points three units to the right and the left of the vertex?

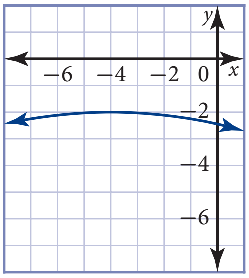
## Practise A

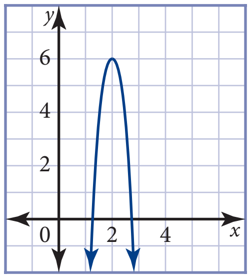
**For help with questions 1 and 2, refer to Example 1.**

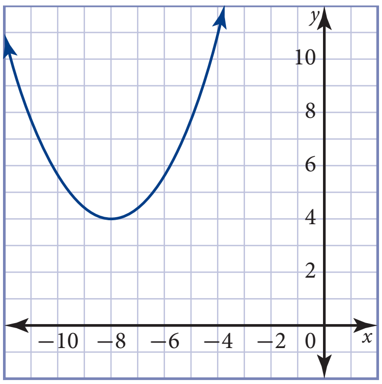
**1.** For each parabola
  **i)** identify the coordinates of the vertex
  **ii)** determine whether $a$ is positive or negative

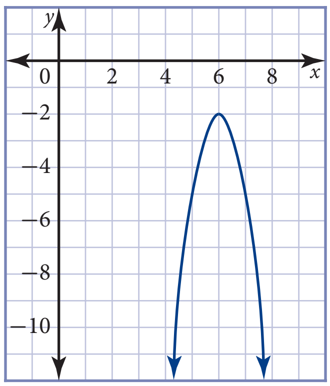
**a)**


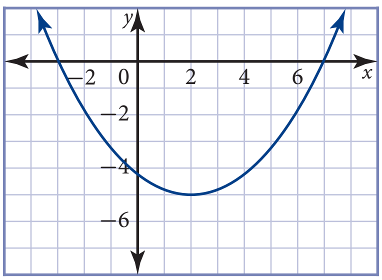

**b)**


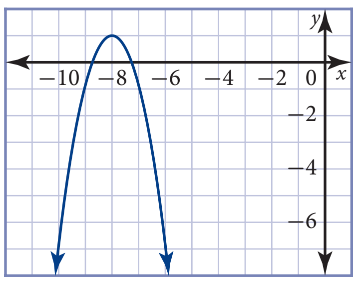

**c)**



**d)**



**e)**



**f)**



**g)**



**h)**


**2.** For each quadratic relation

**i)** identify the coordinates of the vertex

**ii)** determine if the parabola opens upward or downward

**iii)** determine if the parabola is vertically stretched or vertically compressed

**iv)** sketch the graph

**a)** $y = 2(x - 3)^2 + 12$

**b)** $y = -0.5(x - 10)^2 - 1$

**c)** $y = -7(x + 4)^2 - 8$

**d)** $y = -(x + 20)^2 - 5$

**e)** $y = 0.5(x - 11)^2 - 3$

**f)** $y = 8(x + 2)^2 + 9$

**g)** $y = -0.5(x + 6)^2 + 7$

**h)** $y = 2(x - 8)^2 + 2$

**i)** $y = 7.5(x + 2)^2 - 1$

**j)** $y = -0.8(x - 4)^2 + 6$

**For help with question 3, refer to Example 2.**

**3.** Graph each quadratic relation by plotting the vertex and two other points. Then, draw a smooth curve through the points.

**a)** $y = 4(x - 6)^2 + 10$

**b)** $y = -0.25(x + 4)^2 - 2$

**c)** $y = 0.8(x + 1)^2 + 9$

**d)** $y = -2.5(x - 9)^2 - 5$

**e)** $y = 0.2(x - 3)^2 - 7$

**f)** $y = 5(x + 7)^2 + 3$

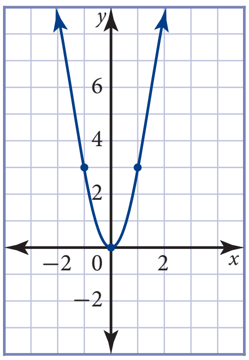
**g)** $y = 6(x + 2)^2 - 1$

**h)** $y = -0.5(x - 10)^2 + 6$

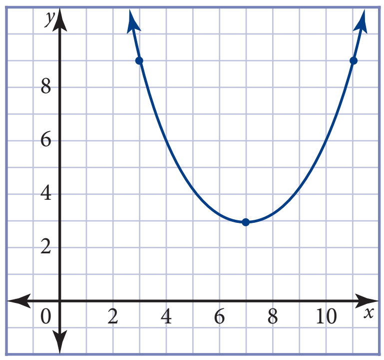
**For help with questions 4 and 5, refer to Example 3.**

**4.** Identify the coordinates of the vertex of each parabola. Then, write an equation for the relation in the form $y = a(x - h)^2 + k$.

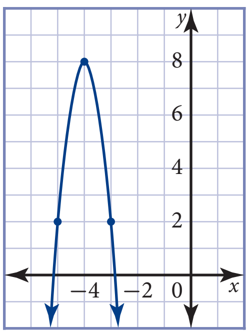
**a)**



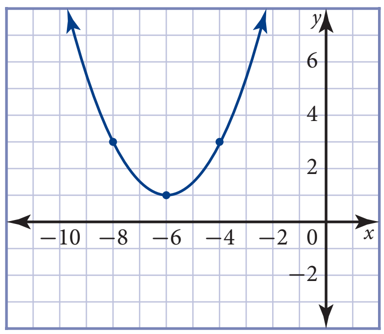

**b)**




**c)**

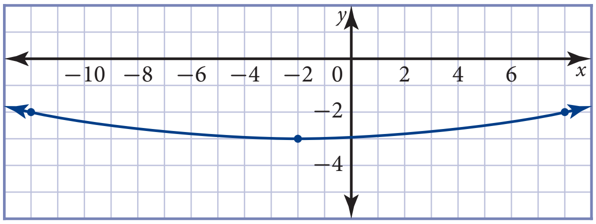



**d)**

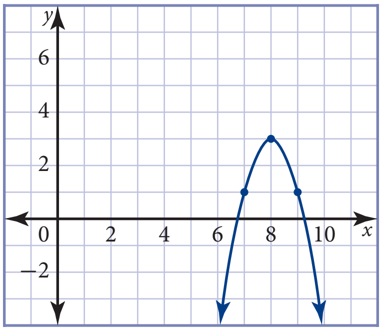


5. Write an equation for each parabola in the form $y = a(x - h)^2 + k$.

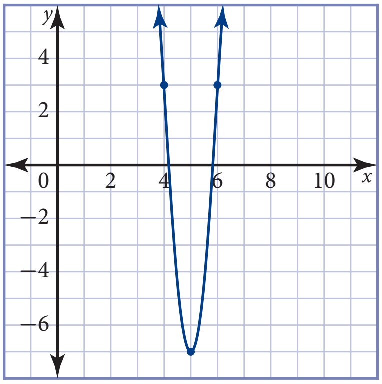
a)



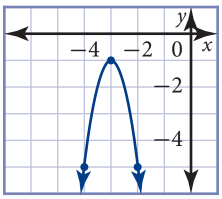

b)




c)




d)




## Apply B

6. A football was kicked. Its path can be modelled by the relation $h = -0.1(d - 8.7)^2 + 7.6$ where $h$ is the football's height above the ground and $d$ is the horizontal distance from where the football was kicked, both in metres.

   a) What is the vertex of the parabola?

   b) What is the football's initial height?

   c) Sketch a graph of the football's path.

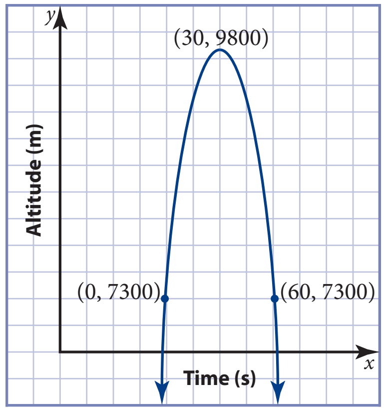
   d) What does the vertex represent in terms of the football's path?

**Literacy Connect**

7. The manager of a hockey arena is pricing tickets for an upcoming game. She knows that if she increases the ticket price she will sell fewer tickets. The situation is modelled by the relation $R = -100(P - 15)^2 + 22\ 500$, where $R$ is the total revenue and $P$ is the ticket price, both in dollars.
   a) Create a table of values and graph the relation.
   b) What is the vertex of the parabola? What do the coordinates of the vertex represent in this situation?

8. "The Vomit Comet" is the nickname of a jet used to simulate zero gravity (0-g) for astronauts. To simulate 0-g, the jet flies in a parabolic arc, starting at an altitude of about 7300 m. After climbing for about 30 s, the jet reaches its maximum altitude at about 9800 m, where the weightless effect occurs. The jet descends back to an altitude of 7300 m after about 60 s and then repeats the process.

**Path of the Vomit Comet**




   a) Write a quadratic relation to model the path of the jet.
   b) The effects of simulated 0-g start being felt at about 20 s. What is the jet's altitude at this time?

**Chapter Problem**

**9.** The next part of your video game models a biker jumping off a ramp at a motocross event. The biker's path can be modelled by the relation $h = -0.03(d - 9.5)^2 + 5$, where $h$ is the biker's height above the ground and $d$ is the biker's horizontal distance from the end of the ramp, both in metres.

**a)** What is the biker's initial height above the ground (when $d = 0$)?

**b)** What is the vertex of the parabola? What information do the coordinates of the vertex give about the biker's position?

**c)** Sketch a graph of the biker's path.

**d)** Use a graphing calculator or graphing software to determine where the biker will land.

### Achievement Check


**10.** Serina has a summer job as a surfing instructor in Tofino, B.C. She usually charges \$40 per lesson and gives about nine lessons per week. One month, she charges different rates each week to see the effect on her revenue. She notices that for every \$5 decrease in the cost of a lesson, two more people sign up for lessons each week. What should Serina charge per lesson, so that her weekly revenue is a maximum?

**a)** Copy and complete the table.

| Charge per Lesson (\$) | Expected Number of People per Week | Weekly Revenue (\$) |
|---|---|---|
| 45 | | |
| 40 | 9 | $(40)(9) = 360$ |
| 35 | 11 | |
| 30 | | |
| 25 | | |
| 20 | | |

**b)** Graph the relation.

**c)** Describe the graph.

**d)** Write a relation of the form $y = a(x - h)^2 + k$ to represent the graph.

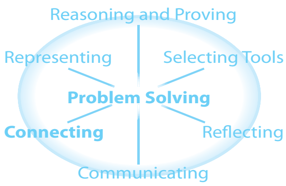
**e)** How much should Serina charge per lesson? Justify your answer.

**f)** Predict how the shape of the parabola and the value of the maximum would change if, for every \$5 decrease in the cost of a lesson, three more people enrolled instead of two more.

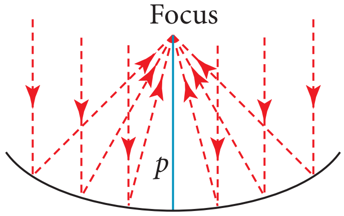
## Extend C

11. The standard form of the equation for a quadratic relation is $y = ax^2 + bx + c$. Go to *www.mcgrawhill.ca/links/foundations11* and follow the links to 4.4. Download the file **4.4 Standard.gsp**. Investigate how changing $a$, $b$, and $c$ affects the graph. Which form of the equation (the standard form, $y = ax^2 + bx + c$, or the vertex form, $y = a(x - h)^2 + k$) is more useful for modelling quadratic relations? Explain your choice.

12. A catapult is a device that was used in ancient times to launch projectiles across a distance. Develop a relation that would show the path of a catapulted boulder that lands 100 m away from where it was launched. Can there be more than one relation? Explain.

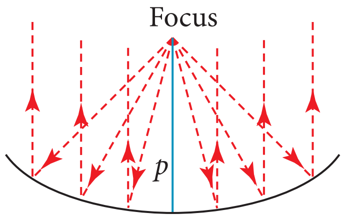



13. The mathematician Archimedes, who lived in the third century B.C.E., was the first to discover the properties of a parabolic reflector. He found that parallel rays coming into a parabolic reflector will reflect to the same spot, called the focus. This property is modelled by the relation $y = \frac{1}{4p}(x - h)^2 + k$, where $(h, k)$ is the vertex of the parabola and $p$ is the distance from the focus to the vertex.




a) Where is the focus on a satellite dish that is 3.0 m across and 0.3 m deep?

b) On a headlight, the process works in reverse. If a bulb is placed at the focus, then the light rays reflect and travel straight out the front of the headlight. For a headlight that is 15 cm wide and 5 cm deep, where should the bulb be placed?



# 4.5 Interpret Graphs of Quadratic Relations

The seventeenth-century mathematician Galileo Galilei was the first to show that projectiles travel in approximately parabolic arcs. This helped the military commanders of his time make more accurate predictions of where cannonballs would land when fired. Now, air relief efforts use the mathematics of projectiles to deliver emergency food and medical supplies to areas suffering from war and natural disasters.

**Example 1**

### Use the Graph and Vertex to Interpret a Quadratic Relation

A construction worker drops his wrench. Its fall is modelled by the relation $h = -4.9t^2 + 342$, where $h$ is the height above the ground, in metres, and $t$ is the time after the wrench was dropped, in seconds.

**a)** Graph the relation. Describe the shape, position, and orientation of the graph.

**b)** How far above the ground was the wrench when it was dropped?

**c)** How far has the wrench fallen after 5 s?

**d)** How far has the wrench fallen after 10 s? How can you interpret the answer realistically?

**e)** When will the wrench hit the ground?

## Solution

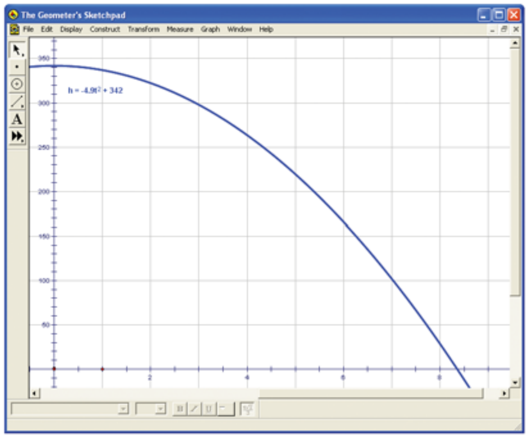
**a)** Use the *The Geometer's Sketchpad®* to graph the relation.

The relation can also be graphed by hand or using a graphing calculator.

- Open a new sketch.
- From the **Graph** menu, choose **Plot New Function**.
- Enter the relation with $x$ as the variable instead of $t$.
- Select **OK**.
- From the **Graph** menu, choose **Grid Form>Rectangular Grid**. Stretch or shrink each scale by dragging the scale points at (1, 0) and (0, 1).

**Technology Tip**
The [^] key (shift 6) is the symbol for an exponent.

**Technology Tip**
To rescale the axes, slide the points on the $x$- and $y$-axes.


The graph is a parabola that opens downward.
$a = -4.9$, so the parabola is vertically stretched relative to the graph of $y = x^2$.
$h = 0$ and $k = 342$, so the vertex is (0, 342).

**b) Method 1: Use the Equation**

When the wrench was dropped, $t = 0$, so substitute 0 for $t$ in the relation:

$$\begin{aligned} h &= -4.9t^2 + 342 \\ &= -4.9(0)^2 + 342 \\ &= 342 \end{aligned}$$

The wrench was 342 m above the ground when it was dropped.

**Method 2: Use the Vertex**

The vertex is a maximum, so $k = 342$ gives the wrench's maximum distance from the ground. The wrench was 342 m above the ground when it was dropped.

**c)** The relation will give a more accurate answer than the graph. To find the distance the wrench has fallen after 5 s, first calculate how far the wrench is above ground when $t = 5$.

$$\begin{aligned} h &= -4.9t^2 + 342 \\ &= -4.9(5)^2 + 342 \\ &= -4.9(25) + 342 \\ &= -122.5 + 342 \\ &= 219.5 \end{aligned}$$

The wrench is 219.5 m above the ground. To find the distance the wrench has fallen, subtract 219.5 m from the wrench's initial height above the ground.

$$\begin{aligned} \text{Distance fallen} &= 342 - 219.5 \\ &= 122.5 \end{aligned}$$

The wrench has fallen 122.5 m after 5 s.

**d)** To calculate the wrench's height above the ground after 10 s, substitute $t = 10$.

$$\begin{aligned} h &= -4.9t^2 + 342 \\ &= -4.9(10)^2 + 342 \\ &= -4.9(100) + 342 \\ &= -490 + 342 \\ &= -148 \end{aligned}$$

The value of $-148$ does not make sense in this situation because it would mean that the wrench was 148 m below the ground after 10 s. The answer could make sense if the wrench fell down a well or into a lake, which would allow for a depth below ground level.

**e)** When the wrench hits the ground, $h = 0$.

$$\begin{aligned} h &= -4.9t^2 + 342 \\ 0 &= -4.9t^2 + 342 \\ 0 + 4.9t^2 &= -4.9t^2 + 342 + 4.9t^2 && \text{Add } 4.9t^2 \text{ to both sides.} \\ 4.9t^2 &= 342 \\ \frac{4.9t^2}{4.9} &= \frac{342}{4.9} && \text{Divide both sides by 4.9.} \\ t^2 &\doteq 69.80 \\ \sqrt{t^2} &\doteq \sqrt{69.80} && \text{Find the square root of both sides.} \\ t &\doteq 8.35 \end{aligned}$$

The wrench hits the ground after approximately 8.4 s.

## Example 2

### Quadratic Model for the Path of a Football

A football player kicks a football held 0.5 m above the ground. The football reaches a maximum height of 30 m at a horizontal distance of 18 m from the player.

**a)** Determine a quadratic relation that models the path of the football.

**b)** At what horizontal distance from the player does the football hit the ground?

### Solution

**a)** Let $y$ represent the height above the ground and $x$ represent the horizontal distance from the player, both in metres.

The football reaches its maximum height of 30 m at a horizontal distance of 18 m, so the vertex is at (18, 30).

$$y = a(x - 18)^2 + 30$$

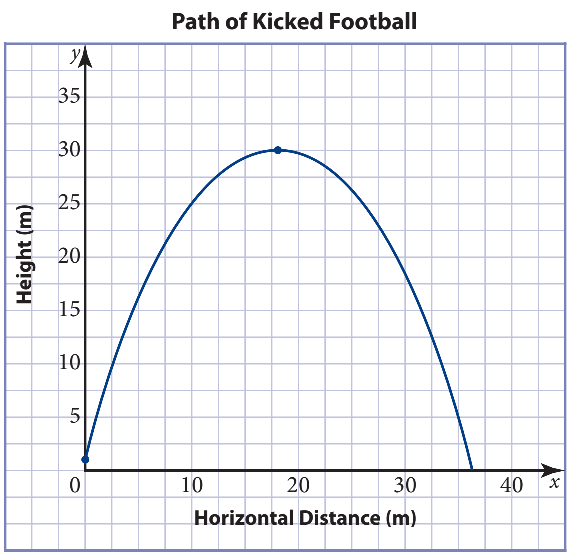
Substitute the coordinates of the vertex into the relation $y = a(x - h)^2 + k$.

$$0.5 = a(0 - 18)^2 + 30$$
$$-29.5 = 324a$$
$$\frac{-29.5}{324} = \frac{324a}{324}$$
$$-0.091 \doteq a$$

The initial position of the football is (0, 0.5). Substitute the coordinates of this point and solve for $a$.

The path of the football can be modelled by the relation $y = -0.091(x - 18)^2 + 30$.

Graph to check that the parabola passes through the points (0, 0.5) and (18, 30).



**b)** To calculate the football's horizontal distance from the player when it hits the ground, let $y = 0$ and solve for $x$.

$$y = -0.091(x - 18)^2 + 30$$
$$0 = -0.091(x - 18)^2 + 30$$
$$-30 = -0.091(x - 18)^2 \quad \text{Subtract 30 from both sides.}$$
$$\frac{-30}{-0.091} = \frac{-0.091(x - 18)^2}{-0.091} \quad \text{Divide both sides by } -0.091.$$
$$329.67 \doteq (x - 18)^2$$
$$\sqrt{329.67} \doteq \sqrt{(x - 18)^2} \quad \text{Find the square root of both sides.}$$
$$18.16 \doteq x - 18 \quad \text{Add 18 to both sides.}$$
$$36.16 \doteq x$$

The football lands about 36.2 m from the player.

## Key Concepts

- When modelling a quadratic relation, choose a convenient location for the origin.
- A quadratic relation can be represented by a table, a graph, or an equation.
- To determine the equation for a quadratic relation from a graph, use the vertex and another point to solve for *a*. Then, write the equation using the vertex and the value of *a*.

### Discuss the Concepts

**D1.** Describe how to write a quadratic relation from its graph.

**D2.** What information do you need to have about a parabola to write its equation?

## Practise A

**For help with questions 1 and 2, refer to Example 1.**

**1.** Find the $y$-intercept for each relation.

**a)** $y = -15x^2 + 25x - 7$

**b)** $y = 0.45x^2 - 0.17x + 20$

**c)** $y = 20(x - 12)^2 + 15$

**d)** $y = -0.5(x + 1.5)^2 + 4.5$

**e)** $y = 10x^2 + 8x - 3$

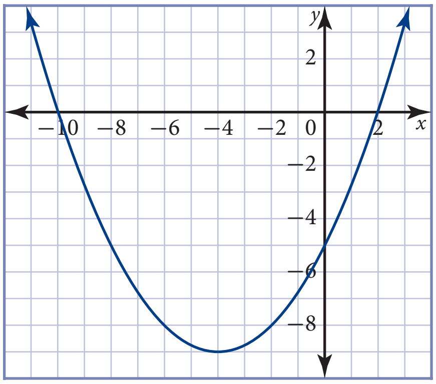
**f)** $y = 0.2(x - 3.4)^2 + 1$

**g)** $y = -0.1x^2 - 0.4x - 1.8$

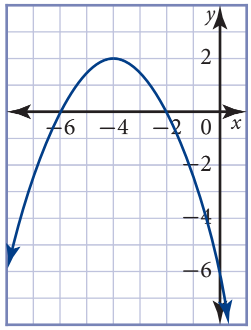
**h)** $y = -3(x + 2)^2 - 9$

**2.** For each parabola, identify
- the $x$-intercepts
- the $y$-intercept
- the maximum or minimum value
- the coordinates of the vertex

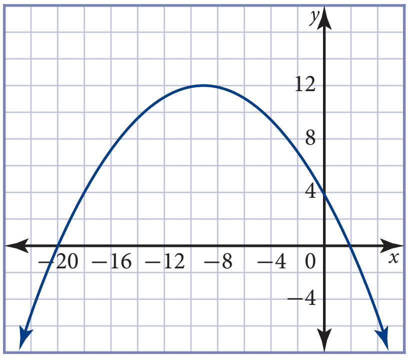
**a)**



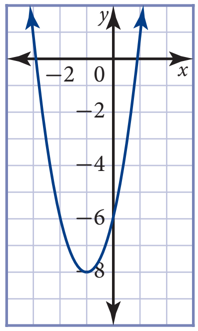

**b)**



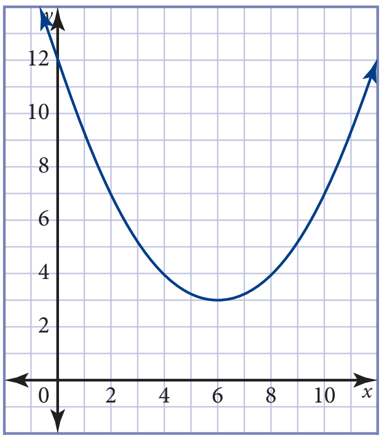

**c)**



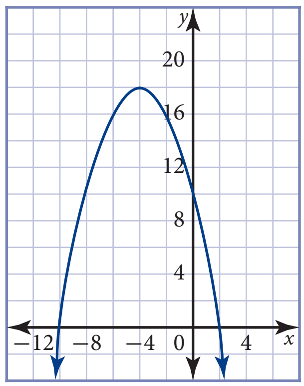

**d)**




**e)**




**f)**



**For help with question 3, refer to Example 2.**

**3.** Sketch each parabola. Start each one at the vertical axis. Then, determine its equation.

**a)** The parabola is 6 cm wide and 27 cm deep.

**b)** The parabola starts at its highest point of 24.5 m. It drops to zero 7 m to the right of the highest point.

**c)** The parabola has a width of 20 and a height of 10.

**d)** The parabola starts at a minimum. It reaches a height of 9375 units at a point 25 units to the right.

## Apply B

**4.** The Windsor–Detroit International Freedom Festival hosts one of the largest fireworks displays in the world. The fireworks are set off over the Detroit River. The path of a particular firework rocket is modelled by the relation $h = -4.9(t - 2)^2 + 169.6$, where $h$ is the rocket's height above the water, in metres, and $t$ is the time, in seconds.

**a)** How long will the rocket take to reach its maximum height? What is the maximum height?

**b)** A firework rocket will stay lit for an average of 5 s. What will the height of a rocket be 5 s after it is launched?

**Literacy Connect**

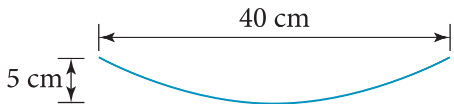
**5.** Car manufacturers have invented some unique car designs to help drivers cope with rising fuel costs. The Smart Car Fortwo is a very small, fuel-efficient vehicle. The distance it travels when accelerating from rest can be modelled by the relation $d = 1.4t^2$. The Tesla Roadster is an electric car with a top speed of over 200 km/h. The distance it travels when accelerating from rest can be modelled by the relation $d = 6.9t^2$. In both cases, $d$ is the distance, in metres, and $t$ is the time, in seconds.

**a)** Construct a table of values for each relation, using $t$-values from 0 to 5. Graph the data on the same set of axes.

**b)** In 5 s, how much farther will the Tesla travel relative to the Smart Car? What parts of the relations indicate the results that you found?

**c)** How far will each car travel in the first second? In the fourth second? What do your answers tell you about the speeds of the cars during each time period?

**6.** The shape of a satellite dish is parabolic. The dish is 5 cm deep and 40 cm wide. Write a relation of the form $y = a(x - h)^2 + k$ that models the shape of this dish. What assumption are you making?




**7.** A projectile is fired straight up from the ground. It reaches a maximum height of 101.25 m after 4.5 s. Then, it falls to the ground 4.5 s later.

**a)** Write a relation that models this situation.

**b)** What is the height of the projectile after 3 s? Is there another time when the projectile is at the same height above the ground? Explain.

**Chapter Problem**

**8.** When studying ballistics, Galileo Galilei found that by changing the angle between a cannon and the ground, a cannonball could be fired different distances using the same amount of explosive.

The same method is used by video game designers when determining the angle of a ramp in a ski jump game. For each relation in the table, $h$ is the ski jumper's height above the ground and $d$ is the ski jumper's horizontal distance from the ramp, both in metres.

| Ramp Angle | Quadratic Relation |
|---|---|
| 25° | $h = -0.018d^2 + 0.47d + 2$ |
| 35° | $h = -0.022d^2 + 0.70d + 2$ |
| 45° | $h = -0.029d^2 + 1.00d + 2$ |
| 55° | $h = -0.044d^2 + 1.43d + 2$ |
| 65° | $h = -0.081d^2 + 2.15d + 2$ |
| 75° | $h = -0.216d^2 + 3.73d + 2$ |

**a)** Use a graphing calculator or graphing software to graph each relation in the table on the same set of axes.

**b)** Which angle gives the greatest horizontal distance?

**c)** Which angle gives the greatest height?

**d)** What is the meaning of the constant 2 in each relation?

## Extend C

**9.** Refer to question 5. The Smart Car starts a race at 0 s. At what time should the Tesla start so that both cars are at the same point after 5 s of the race? Justify your answer.

# 4 Review

## 4.1 Modelling With Quadratic Relations, pages 168–179

**1.** Is each relation quadratic? How do you know?

**a)** $y = 3x - 15$

**b)** $y = 4x^2 - 2x + 8$

**c)**

| x | y |
|---|---|
| −5 | 1 |
| 0 | 4 |
| 5 | 16 |
| 10 | 64 |
| 15 | 256 |

**2.** A baseball is thrown upward. The path of the ball is modelled by the relation $h = -4.9t^2 + 15t + 2$, where $h$ is the baseball's height above the ground, in metres, and $t$ is the time, in seconds.

**a)** Copy and complete the table.

| Time (s) | Height (m) |
|---|---|
| 0.0 | |
| 0.5 | |
| 1.0 | |
| 1.5 | |
| 2.0 | |
| 2.5 | |
| 3.0 | |

**b)** How long will it take the baseball to reach its maximum height?

**c)** After how many seconds will the baseball land?

**d)** How can you tell that this relationship is quadratic? List as many reasons as possible.

## 4.2 The Quadratic Relation $y = ax^2 + k$, pages 180–193

**3.** Describe the shape and position of each parabola relative to the graph of $y = x^2$.

**a)** $y = x^2 - 3.4$

**b)** $y = -0.35x^2$

**c)** $y = 0.005x^2 + 15$

**d)** $y = 6.5x^2 - 3.4$

**4.** Sketch the graph of each relation in question 3.

**5.** Write a relation that models each table of values.

**a)**

| x | y |
|---|---|
| −1 | −88 |
| 0 | −100 |
| 1 | −88 |
| 2 | −52 |
| 3 | 8 |
| 4 | 92 |

**b)**

| x | y |
|---|---|
| −5.0 | 19.5 |
| 0.0 | 20.0 |
| 5.0 | 19.5 |
| 10.0 | 18.0 |
| 15.0 | 15.5 |
| 20.0 | 12.0 |

## 4.3 The Quadratic Relation $y = a(x - h)^2$, pages 194–203

**6.** Write a relation that models each table of values.

**a)**

| x | y |
|---|---|
| 8 | −32 |
| 10 | 0 |
| 12 | −32 |
| 14 | −128 |
| 16 | −288 |
| 18 | −512 |

**b)**

| x | y |
|---|---|
| −26 | 60 |
| −16 | 15 |
| −6 | 0 |
| 4 | 15 |
| 14 | 60 |
| 24 | 135 |

### 4.4 The Quadratic Relation $y = a(x - h)^2 + k$, pages 204–217

**7.** Describe the shape and position of each parabola relative to the graph of $y = x^2$.

**a)** $y = -0.004(x - 18)^2 + 15$

**b)** $y = 7(x + 1)^2 - 2$

**c)** $y = -80(x + 9)^2 + 10.8$

**d)** $y = 0.6(x - 40)^2$

**8.** Sketch the graph of each relation in question 7.

**9.** A computer repair technician is deciding what hourly rate to charge for her services. She knows that if she charges \$60/h, she will get 30 h of work per week. She also knows that for every \$5 increase in her hourly rate, she will lose 4 h of work per week.

**a)** Copy and complete the table.

| Hourly Rate (\$) | Expected Number of Hours per Week | Weekly Revenue (\$) |
|---|---|---|
| 45 | | |
| 50 | | |
| 55 | | |
| 60 | 30 | (60)(30) = \$1800 |
| 65 | 26 | |
| 70 | | |

**b)** Graph the relation between hourly rate and weekly revenue.

**c)** Write a relation in the form $y = a(x - h)^2 + k$ to represent the graph.

**d)** What hourly rate should the technician charge to earn the maximum weekly revenue?

**10.** Sketch the graph of each parabola. Then, determine its equation.

**a)** opens upward, vertex is $(3, -5)$, passes through point $(13, 20)$

**b)** opens downward, vertex is $(-4, 7)$, passes through point $(0, -39)$

### 4.5 Interpret Graphs of Quadratic Relations, pages 218–225

**11.** One of the largest solar furnaces in the world is in Odeillo, France. The parabolic mirror is 54 m wide and 10 m deep. Write a relation to model the parabolic shape of the mirror.

**12.** A water balloon is thrown upwards. The balloon follows a path modelled by the relation $h = -2.6d^2 + 7.8d + 2.15$, where $h$ is the balloon's height above the ground and $d$ is the balloon's horizontal distance from the release point, both in metres.

**a)** Copy and complete the table. Graph the relation.

| ***d*** | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| ***h*** | | | | | | | |

**b)** What was the balloon's initial height above the ground?

**c)** Write a relation in the form $y = a(x - h)^2 + k$ to model the balloon's path.

# 4 Practice Test

**For questions 1 to 6, choose the best answer.**

**1.** Which of these relations is quadratic?

**A** $y = 0.5x - 7$

**B** $y = 5.8x + 3x^2 - 9$

**C** $y = 4x^3 + 2x^2 - 5x + 1$

**D** $3x + 2y + 10 = 0$

**2.** Which of these relations is not quadratic?

**A** the path of a ball thrown in the air

**B** the distance a car travels when it is accelerating

**C** the distance travelled when running at a constant speed

**D** the shape of a satellite dish

**3.** Which parabola has its vertex 3 units above the $x$-axis?

**A** $y = 3(x - 5)^2 + 4$

**B** $y = 5(x + 4)^2 - 3$

**C** $y = 0.1(x - 15)^2 + 3$

**D** $y = 0.3(x + 3)^2 - 10$

**4.** Which parabola has its vertex farthest from the $y$-axis?

**A** $y = 3(x - 5)^2 + 4$

**B** $y = 5(x + 4)^2 - 3$

**C** $y = 0.1(x - 15)^2 + 3$

**D** $y = 0.3(x + 0.8)^2 - 10$

**5.** Which parabola is the most vertically stretched?

**A** $y = 3(x - 5)^2 + 4$

**B** $y = 5(x + 4)^2 - 3$

**C** $y = 0.1(x - 15)^2 + 3$

**D** $y = 0.3(x + 0.8)^2 - 10$

**6.** The parabola represented by the relation $y = -8(x + 15)^2 + 12$ has which vertex?

**A** $(-8, 15)$

**B** $(-15, 12)$

**C** $(-15, -12)$

**D** $(-8, -12)$

**7.** What is always true about the first and second differences of a quadratic relation?

**8.** Describe the shape and position of each parabola relative to the graph of $y = x^2$. Sketch a graph of each parabola.

**a)** $y = 0.5(x + 8)^2$

**b)** $y = -8x^2 - 14$

**c)** $y = -10(x - 7)^2 - 13$

**d)** $y = 0.002(x + 20)^2 + 16$

**9.** A soccer ball is kicked from ground level. When it has travelled 35 m horizontally, it reaches its maximum height of 25 m. The soccer ball lands on the ground 70 m from where it was kicked.

**a)** Model this situation with a relation in the form $y = a(x - h)^2 + k$.

**b)** What is the soccer ball's height when it is 50 m from where it was kicked?

## Chapter Problem Wrap-Up

Throughout this chapter you have developed relations to model the paths of a snowboarder, a skateboarder, a mountain biker, a motocross biker, and a ski jumper. There are many more extreme sports that involve flying through the air. Think of a different sport and develop a relation that can model the sport's motion to complete your video game.

**10.** Pennies are stacked in a triangular pattern.

**a)** Continue the pattern. Copy and complete the table for the first ten layers of pennies.

| Number of Layers | Total Number of Pennies |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |
| 4 | |
| 5 | |

**b)** Describe the relationship between the number of layers and the total number of pennies.

**c)** How many layers are in a triangle made of 105 pennies?

**d)** How many pennies are needed for a triangle with 50 layers? Explain how you found your answer.

**11.** A basketball was thrown upward. The basketball's path is given by the relation $h = -0.2(d - 2.5)^2 + 4.25$, where $h$ is the basketball's height above the ground and $d$ is the basketball's horizontal distance from where it was thrown, both in metres.

**a)** What was the basketball's initial height above the ground?

**b)** What was the basketball's greatest height above the ground? What was the basketball's horizontal distance at this point?

**c)** The basketball was thrown toward a net 6 m away and 3 m above the ground. Will the basketball go through the net? Justify your answer.